Comparing the first mathematics textbook with a modern one: How some famous geometry theorems have been illustrated over time

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1 Introduction

An important component of the instructional value of a textbook is its illustrations. A distinguishing feature of modern mathematics textbooks is the abundance of their illustrations. [1] However, illustrations in a textbook can be relevant or irrelevant. In geometry textbooks relevant illustrations are closely related to the explanation of a mathematical concept or the method of proof adopted by the author. Hence, illustrations in mathematics textbooks should not be an afterthought, but rather an integral part of the text. The production of proper mathematical illustrations is not a simple task. [2] Santos [3] found that even though the illustration in a mathematics test was just cosmetic, almost half of the students used it nonetheless. Bauman [4] noted figures in a science textbook that were inconsistent with the text of the book. Parzysz [5] described geometrical illustrations that lead to misconceptions because they contained many implicit conventions.

[1] Santos-Bernard, 1996, 251
[2] Lo Bello, 2003b, 236
[3] Santos-Bernard, 1996, 256
[4] Bauman, 1992
[5] Parzysz, 1991

Dowling [6] performed a sociological analysis of two mathematics textbooks used in the United Kingdom. The author found a strong correlation between the way that illustrations were made and the proficiency level of the students for whom the textbook was written. Dowling [7] described how the illustration reflected the intended career path of the students, whether it would be intellectual or technical. For instance, the researcher found that in the case of an intended intellectual orientation the illustration provided a third party, distanced and abstracted view of the topic. On the contrary, the intended technical orientation presented the mathematical topic at a mundane, procedural, and first-person level.

The majority of comparative textbook illustration studies have been latitudinal, across countries. For instance, a comparison between national educational systems by Mayer [8] showed that U.S. mathematics textbooks contain fewer relevant illustrations than comparable Japanese textbooks. However, there is very little in the literature about comparative studies that are longitudinal, across time, on the illustrations of mathematics textbooks.

Based on the assuption that in geometry textbooks the illustrations are a reflection of the explanation and method of proof adopted by the author, we decided to analyze a set of geometry theorems from *The Elements* of Euclid (Table

2 Materials and Methods

As a conpemporary reference we used a modern high school textbook, *Geometry*, a product of the Integrated Mathematics Project of the University of Chicago[9]. This textbook has been adopted in the state of Texas. For a description of the University of Chicago *Integrated Mathematics Project* see Senk[10]. We compared this textbook with the first geometry textbook ever written, *The Elements* [11] in its original Greek a few of its old translations. We also looked at a few commentaries of *The Elements*.

The Elements, Stocheía, of Euclid, Eukleídēs, has been one of the longest used textbooks and is certainly the most famous mathematics textbook in history. It was written in Alexandria about 300 BCE in the Greek language (Heath, 2002, p. ix). Afterwards it has been commented and translated into many other languages. The

[6] Dowling, 1996
[7] Dowling, 1996
[8] Mayer et al., 1995
[9] Usiskin et al., 1998
[10] Senk, 2003
[11] Fitzpatrick, 2007

first known revision was by Theon in Greek and the first known commentary was by Proclus, also in Greek. The first known translation to another language was into Latin done by Boethius about 500 CE (Busard, 2005, p. 1). This Latin version was soon lost except for a few fragments that have survived.

Abū'l-'Abbas al-Faḍl ben Ḥatim al-Nayrīzī, a Persian mathematician and astronomer, wrote a popular commentary on *The Elements* during the reign of the Caliph al-Mutadid (892-902). This commentary was based on the text of *The Elements* as translated by al-Hajjāj into Arabic. He added his comments to the text as well as comments from Simplicius and Heron[12]. We used the drawings from LoBello[13] where he replaced the Arabic letters indicating vertexes with Greek and Latin letter according to a scheme explained on page 79 of his book.

Albertus Magnus wrote a commentary on Book I of *The Elements* in Latin around 1250-1262 [14]. His source was the translation from Arabic to Latin of the commentary of al-Nayrīzī which was performed by Gerard of Cremona [15]

The first mathematics textbook ever printed was a Latin version of *The Elements.* It was printed in Venice in 1482 [16]. This version was used for a couple of centuries at European universities. The first translation into a modern language was performed by the Italian mathematician Tartaglia [17]. He published in Venice a translation plus commentary in Italian.

Table

We have not been able to find any copy of an Arabic version of *The Elements* in printed form. While there are several articles that describe and comment the Arabic versions of the text and its commentaries, it has been so far impossible for us to obtain a copy of those manuscripts.[18]

We examined at representative set of five theorems[19] from Book I.[20] The list of theorems is based on Allen [21]. Note that numbering is according to *The Elements*,

[12]Lo Bello, 2003a, 24

[13]Lo Bello, 2003a

[14]Lo Bello, 2003b, xv

[15]Lo Bello, 2003b, xxvii

[16]Busard, 2005, Preface

[17]Tartaglia, 1565

[18]There is an on-line resource that provides scans of medieval Arabic mathematical texts hosted by Brown University Leichter (2008).

[19]Euclid names them *Propositions*

[20]*The Elements* is comprised of 13 books

[21]Allen, 2008

Date	Author	Comment
300 BC	Euclid	Original version
60	Heron	Revised edition, lost
300	Pappus	Commentary, lost
400	Theon	Version
500	Boethius	Latin translation, lost
800	al-Ḥajjāj	First Arabic translation
850	Ṭābit bin Qurra	Arabic version
900	al-Nairīzī	Arabic commentary on Book I
910	Isḥāq bin Ḥunain	Major Arabic version
1120	Adelhard	Major Arabic to Latin translation
1255	Albertus Magnus	Latin commentary of Book I
1260	Campanus of Novara	Major Latin edition
1175	Gerard of Cremona	Arabic to Latin translation
1482	Campanus of Novara	First printed edition
1565	Nicoló Tartaglia	Greek to Italian translation
1570	Sir Henry Billingsley	First English translation
1607	Matteo Ricci	First Chinese translation
1883	Heiberg	Critical Greek edition
1908	Heath	Major English edition

Table 1: Versions of *The Elements*

while the naming follows *Geometry* [22] and Allen [23].

We counted the components, that is the segments, arcs and number of drawings, of the relevant illustrations. We counted as segments, straigth lines even if they crossed each other. We did so to perform a quasi-quantitative assessment of the complexity of the illustrations.

3 Results

Here we describe the illustrations of the set of theorems and how they relate to their proofs.

[22] Usiskin et al., 1998
[23] Allen, 2008

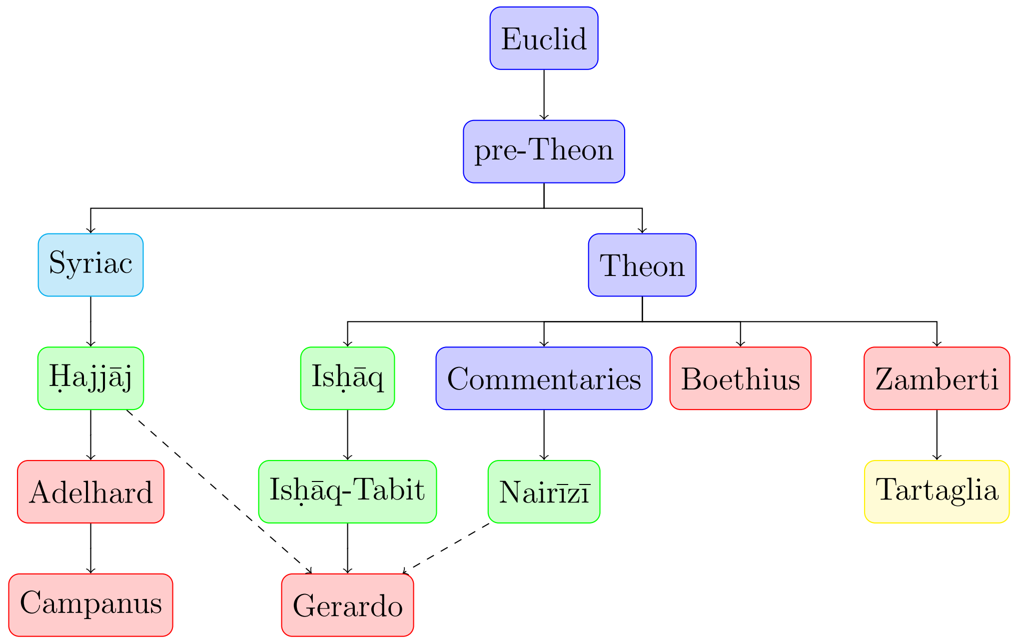

Figure 1: Simplified relationships between Manuscripts.

Proposition I.4

The Greek text of *The Elements* uses the concept of application of one figure to another one. The illustration only shows two equal triangles and the base segment of the second triangle is highlighted by drawing a curve from one end to the other one of the segment. The Heath [24] translation text reads:

> If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

The illustration shown in figure

Euclid used superposition to proof this theorem and an argument from contradiction. Euclid states that two straight lines can not encompass an area and thus $B\Gamma$ has to match EZ. The Greek text does not state why. However, this is an

[24]Heath, 2002, 5

Number	Name
	Side-Angle-Side Congruence Theorem
	Isosceles Triangle Base Angles Theorem
	Isosceles Triangle Base Angles Converse Theorem
	Alternate Interior Angles Theorem
	Pythagorean Theorem

Table 2: Chosen Book I theorems

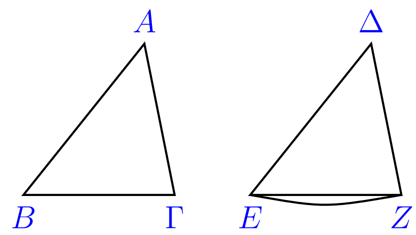

Figure 2: Proposition I.4, Euclid

implication of the First Postulate. Artmann [25] noted that the proof is not complete since the concept of superposition is not part of his axioms. Modern mathematics postulates types of rigid movements of the plane to complete the proof. We will see below that Usiskin et al. [26] use a different strategy to prove this theorem.

The commentary of al-Nayrīzī offers an illustration as shown in figure

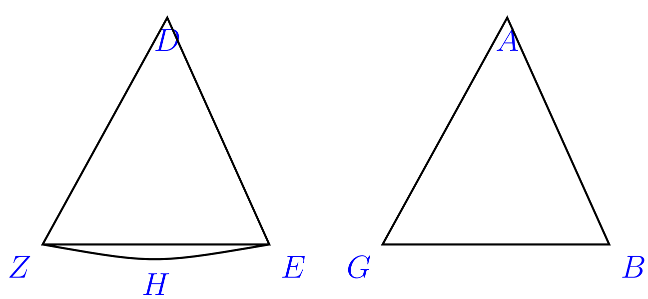

Figure 3: Proposition I.4, al-Nayrīzī

al-Nayrīzī uses the proof by superposition just as the Euclid did originally. However, he adds a proof by contradiction to prove that the two bases are equal. To

[25] Artmann, 1999, 22
[26] Usiskin et al., 1998, 372

do so he states that two straight lines can not enclose a rectilinear surface. This statement refers to a peculiar "sixth postulate," (p. 97). He attempts then to proof it (p. 97-98).

The commentary of Albertus Magnus (Lo Bello, 2003b, p. 44) presents a completely different type of illustration (figure

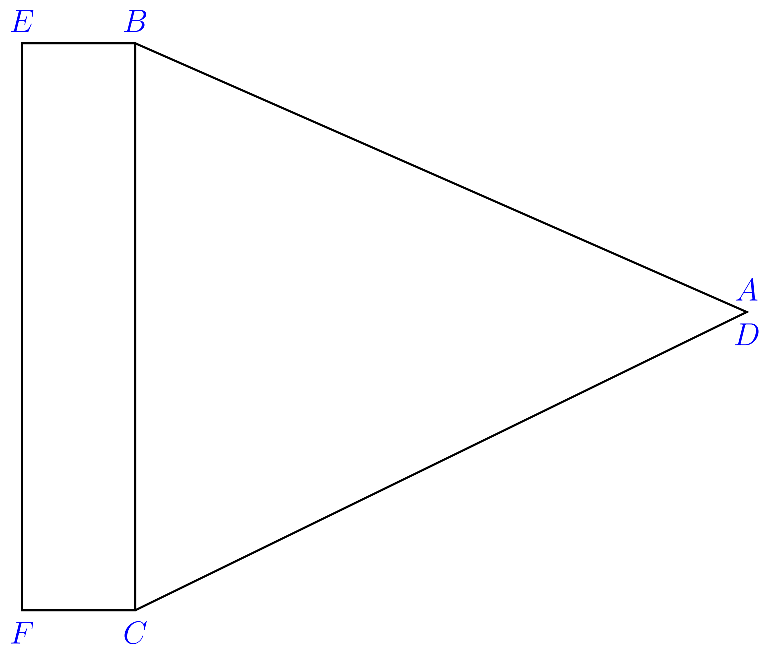

Figure 4: Proposition I.4, Albertus Magnus

Campanus of Novara (Busard, 2005, p. 62) illustrated this theorem as shown in figure

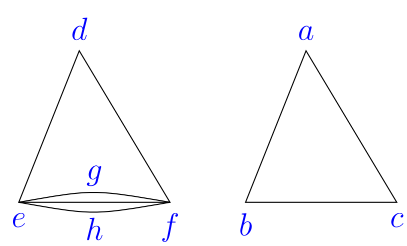

Figure 5: Proposition I.4, Campanus of Novara

Like Euclid and al-Nayrīzī, Campanus uses a proof by superposition. The Latin text reads *Superponam triangulum a b c super triangulum d e f* . . . .

The edition of *The Elements* by Gerard of Cremona (Busard, 1984, p. 6) has just two very simple triangles drawn side by side without any markings other than the vertexes (Figure

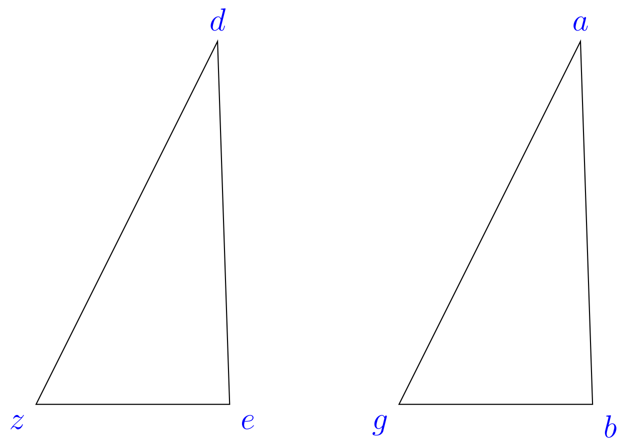

Figure 6: Proposition I.4, Gerard of Cremona

Again, the proof is by superposition. The text reads *Quia cum superposuerimus triangulum abg triangulo dez, et posuerimus punctum a super punctum d* . . . .

The illustrations of the translation and commentary of Tartaglia consist of three panels. The first panel contains two triangles identical to the Campanus edition, but in proper order, the *abc* triangle to the left of the *cde* one (Figure

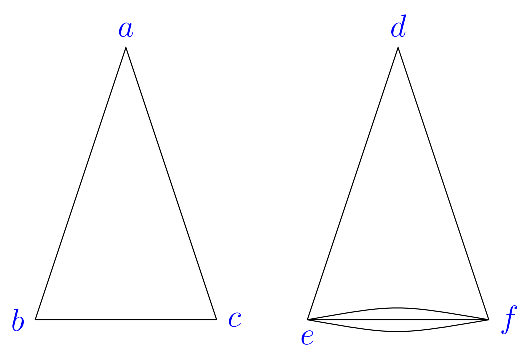

Figure 7: Proposition I.4 A, Tartaglia

The proof by Tartaglia also uses superposition. The text reads *laqual cosa si approba mettendo mentalmente il triangolo .a.b.c. sopra il triangolo .d.e.f. talmente che l'angolo .a. caschi sopra all'angolo .d. et il lato .a.b. sopra il lato .d.e. & il lato .a.c. sopra il lato .d.f. ...*.

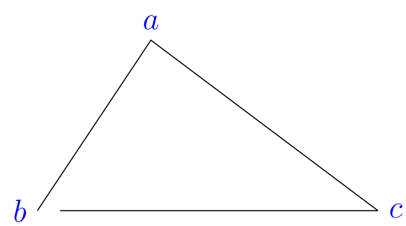

Figure 8: Proposition I.4 B, Tartaglia

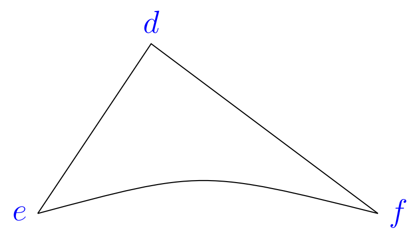

Figure 9: Proposition I.4 C, Tartaglia

The two extra drawings were used by Tartaglia to illustrate a proof by contradiction that uses a sixth postulate, not present in the original *The Elements*, but that is present in the al-Nayrīzī commentary. This postulate he attributes to Simplicius, in Greek Simplíkios of Cilicia, c. 490 - c. 560 CE. The text of this postulate is *two straight lines do not enclose a surface* (Lo Bello, 2003a, p. 97).[27] Campanus of Novara and Gerard of Cremona do not have this sixth postulate. al-Nayrīzī gives a proof of this postulate by contradiction, which contradicts that in mathematics postulates are assumed to be true and thus do not need a proof. Tartaglia (pp. 20-21) does likewise, but uses different illustrations. The text seems to suggest that this is actually not a proof. He seems to appeal to common sense instead of providing a rigorous proof.

[27] More precisely, two segments with common end points

It is interesting to observe that Tartaglia names Proposition I.4 the "first" theorem, *Theorema prima* (p. 28). Indeed it has been observed that Proposition I.4 is actually the first one which is not just a construction and is the first of the basic congruence theorems (Artmann, 1999, p. 21).

The Integrated Mathematics Project textbook names Proposition I.4 the *SAS Congruence Theorem* (Usiskin et al., 1998, p. 372). The book states the theorem as:

> If in two triangles, two sides and the included angle of one are congruent to two sides and the included angle of the other, then the triangles are congruent.

The proof depends on isometric translation mapping 1998, p. 225, Isosceles Triangle Symmetry Theorem 1998, p. 310, and the Transitive Property of Congruence 1998, p. 251. The illustration consists of two congruent triangles where a reflected triangle is constructed below the second one.

The illustration 1998, p. 372 is very similar to the illustration of Proposition VI.5 in the original *The Elements* (Fitzpatrick, 2007, p.161). The main difference is that here as well in all modern geometry illustrations the authors have added ticks to show congruence of sides and arcs for equiangularity (Figure

Table

Edition	S	A	L	D
Euclid	6	1	6	2
al-Nayrīzī	6	1	7	2
Albertus	6	0	6	1
Campanus	6	2	8	2
Gerard	6	0	6	2
Tartaglia	11	4	12	4
IM	8	3	9	2

Table 3: Components of the illustrations, I.4

Proposition I.5

The Greek text reads:

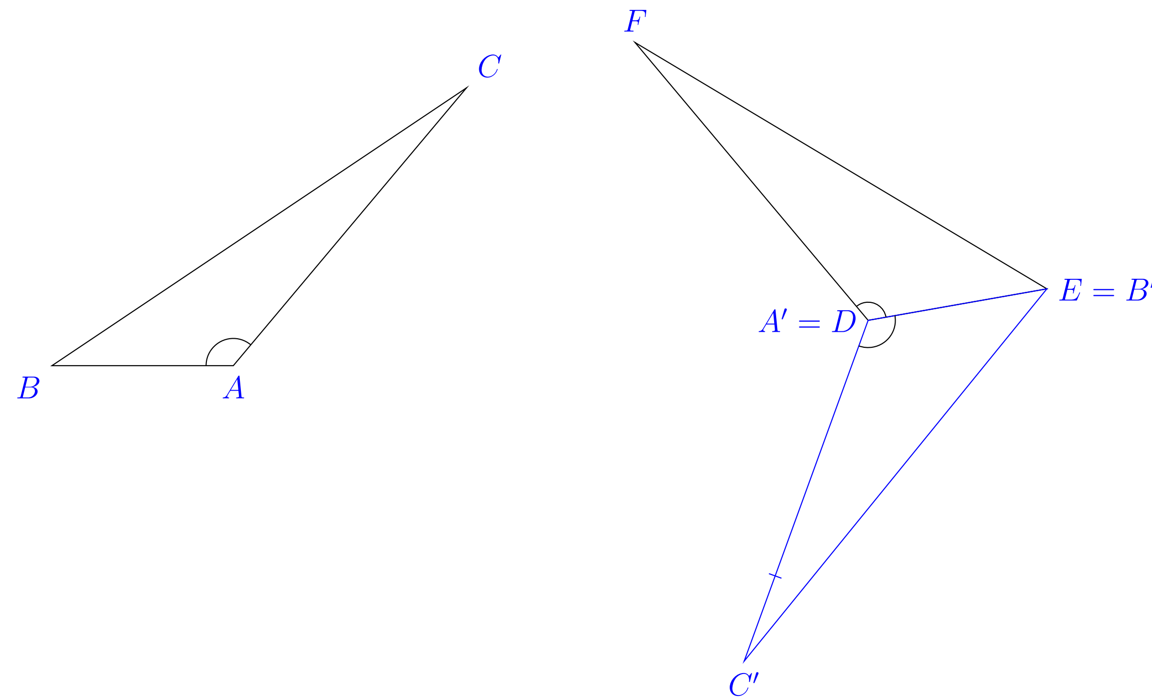

Figure 10: Proposition I.4, Integrated Mathematics

> In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, the angles under the base will be equal to one another (Heath, 2002, p. 5).

The Elements uses as proof the construction two triangles under the base segment. An illustration showing this construction is given in figure

Euclid proofs this theorem by first showing that $\triangle AFC$ is congruent to $\triangle AGB$, then that $\triangle FBC$ is congruent to $\triangle GCB$ using SAS. Then he shows that FBC is equal to BCG and then in a few steps involving angle subtraction shows that $\angle ABC$ is congruent to $\angle ACB$ (p. 6).

al-Nayrīzī states the fifth theorem identically to Euclid's version with slightly different wording (Lo Bello, 2003a, p. 114). The proof was illustrated as in Figure

The proof of this commentator follows the one by Euclid, but then he adds an illustration (Figure (

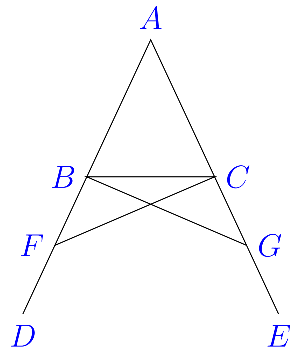

Figure 11: Proposition I.5, Euclid

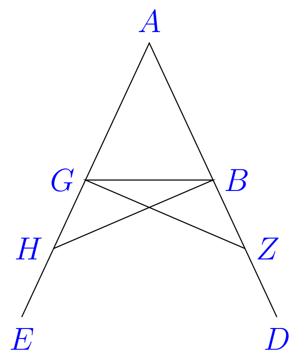

Figure 12: Proposition I.5 A, al-Nayrīzī

Albertus Magnus follows al-Nayrīzī by offering both versions of the proof (Lo Bello, 2003b, p. 45). The first illustration is different from both the Greek and the Arabic versions, see figure

The proof follows the usual path of using SAS and then the subtraction of congruent angles (p. 46). The second illustration is almost identical to the one of al-Nayrīzī, except for the lettering (Figure

Campanus of Novara states the theorem the same way that the Greek original does, stating that **both** upper and lower angles are congruent. The proof is performed by the usual SAS and angle subtraction (Busard, 2005, p. 63). The illustration is very similar to the one by Albertus Magnus with the lettering of the two sides being transposed. Thus sides *ab* and *ac* are extended, *Protractis* in the Latin text (Figure

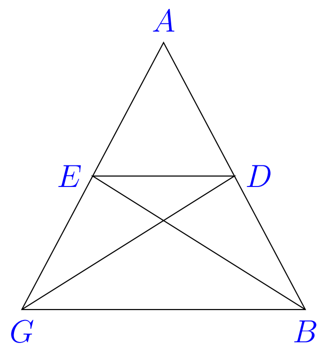

Figure 13: Proposition I.5 B, al-Nayrīzī

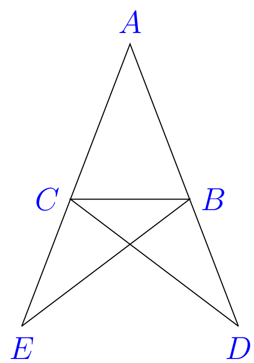

Figure 14: Proposition I.5 A, Albertus Magnus

In the edition of Gerard of Cremona the Latin text of the thesis of the theorem is very similar, but not identical, to the text of Campanus of Novara. The illustration is very similar to the one produced by al-Nayrīzī including the labeling of the points. However, the left extension is longer that the right one (Figure

We propose that Gerard of Cremona drew the unequal lengths of ad and ae to underline the fact that their lengths do actually not matter in the construction. We base this on his statement that point z can be placed randomly between b and d, *Super lineam igitur bd notabo punctum quocumque casu acciderit, sitque punctum illud z*. The proof follows Euclid and the alternate proof is not given.

Tartaglia, in line with the previous theorem, names this theorem 2, *Theorema .2. Propositione .5.* (p. 29). The statement of the theorem, like the original, states that both upper and lower angles are pairwise congruent. The proof is the usual, however, Tartaglia explicitly states that the SAS Theorem needs to be used. That

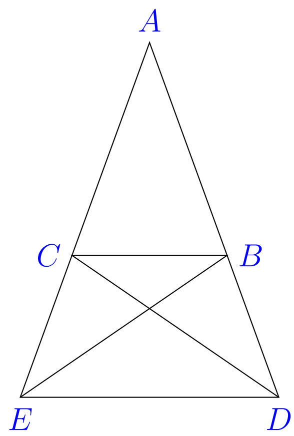

Figure 15: Proposition I.5 B, Albertus Magnus

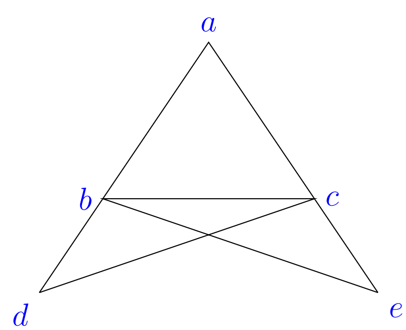

Figure 16: Proposition I.5, Campanus of Novara

has not been done previously. The theorem is illustrated by four figures (figures

Also, Tartaglia draws separately the constructed triangles *acd* and *ebc*. None of the previous authors did so. We explain this unprecedented emphasis on the fact that Tartaglia was a mathematics teacher for most of his life. Indeed Tartaglia wrote at bottom the front page of his book the following two paragraphs:

CON VNA AMPLA ESPOSITIONE

dello stesso traduttore di nuova aggiunta.

TALMENTE CHIARA, CHE OGNI MEDIOCRE

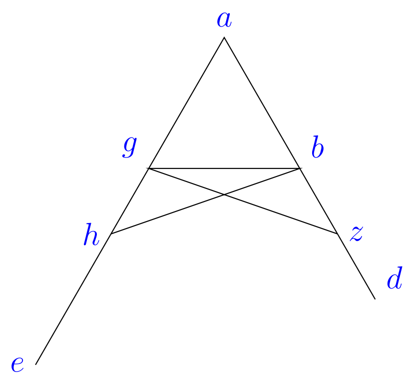

Figure 17: Proposition I.5, Gerard of Cremona

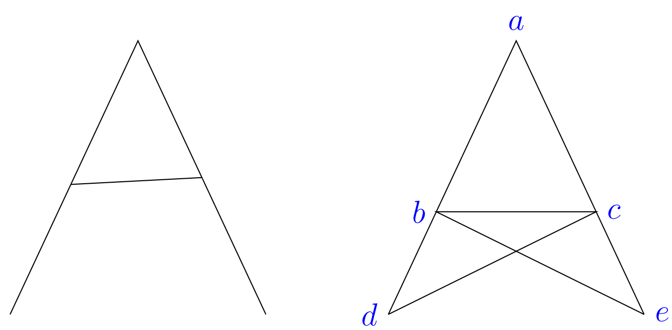

Figure 18: Proposition I.5 A, Tartaglia

ingegno, senza la notitia, ouer suffragio di alcun'altra scientia
con facilitá será capace a porterlo intendere

We translate this as "With an ample explanation by the same translator as a new addition. So clear that any mediocre intellect will be able to understand without the need of any additional knowledge." Whether we share or not the confidence in his pedagogical skills, we can certainly detect an clear intent to make his textbook as clear as possible. No alternative proof was offered by Tartaglia.

The authors of the Integrated Mathematics textbook (Usiskin et al., 1998, p. 310) name this the *Isosceles Triangle Base Angles Theorem*. The proof uses the concepts of symmetry and reflection as developed in an antecedent theorem, the Symmetric Figures Theorem (p. 304) which is not present in *The Elements*. The illustration consists of a single triangle that is traversed by a line of symmetry (Figure

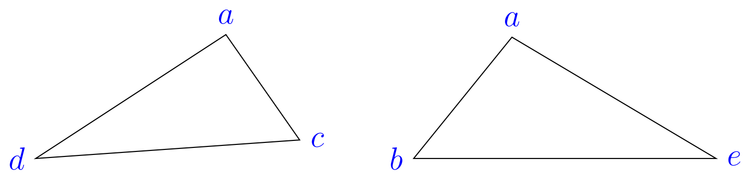

Figure 19: Proposition I.5 B, Tartaglia

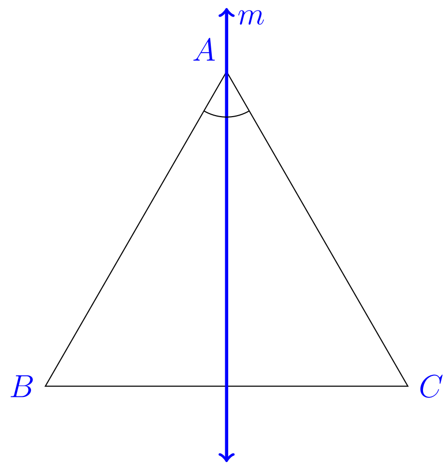

Figure 20: Proposition I.5, Integrated Mathematics

> If a triangle has two congruent sides, then the angles opposite them are congruent.

As proof the text simply considers isosceles triangles to be a specific case of the generic Symmetric Figures Theorem.

Table

Proposition I.6

The Greek text reads:

> If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another (Heath, 2002, p. 6).

Edition	S	A	L	D
Euclid	5	0	7	1
al-Nayrīzī	11	0	12	2
Albertus	11	0	10	2
Campanus	5	0	5	1
Gerard	5	0	7	1
Tartaglia	14	0	11	4
IM	4	2	4	1

Table 4: Components of the illustrations, I.5

The Elements uses a proof by contradiction by assuming that the two sides are not equal. The single illustration that accompanies this proof reflects this approach (Figure

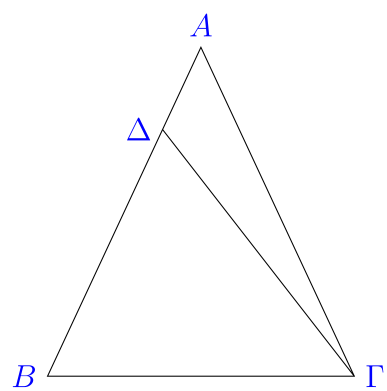

Figure 21: Proposition I.6, Euclid

The illustration drawn by al-Nayrīzī is basically identical to the Greek one, but like the illustration of proposition I.4, he inverts the drawing (Figure

As with the previous proposition, the commentator adds another proof for the same theorem. This proof constructs lower triangles and steps through demonstrations of congruent triangles and subtractions of segments (Figure

Notice that the labeling of the vertexes was changed from the previous drawing. We deduce that this happened because al-Nayrīzī based this drawing on the previous figure

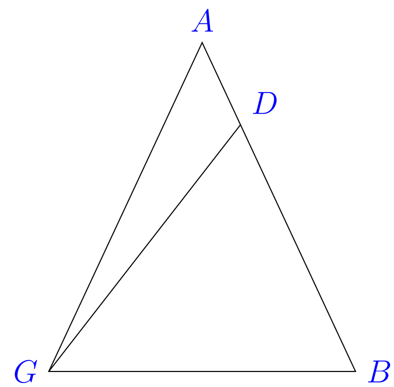

Figure 22: Proposition I.6 A, al-Nayrīzī

Albertus Magnus notes that his proposition is the converse of I.5 and proposes that they should be combined in a single statement which he then states (Lo Bello, 2003b, p. 48). As al-Nayrīzī before him, he first gives the conventional proof and then an alternate one which follows the Persian commentator. Albertus illustrated both proofs as shown in Figures

The first figure has confusing labeling. Notice how the letter C is skipped. In the Greek and Arabic alphabets the third letter are Γ and *Jim*. We suppose that Albertus compromised between fidelity to his sources and the Latin alphabet. Since he could not place a C because it was not in his Arabic manuscript(s) and could not place a G because that would be out of order in the Latin alphabet, he just moved to the next available letter, D. He does not do so in his second illustration, where he follows the Latin alphabet from A to H. I suppose he did depart from his source to avoid confusion.

The text by Campanus (Busard, 2005, p. 64) was illustrated as by Figure

Gerard of Cremona follows the usual proof. His illustration (Figure

Tartaglia names Proposition I.6 as Theorem 3. The text reads "Theorema .3." (p. 30). The illustration consists of one drawing which is equal to the original text (Figure

Tartaglia notes that this theorem is the converse of the preceding one, "conuerso della precedente ..." The commentator/translator uses a proof by contradiction and

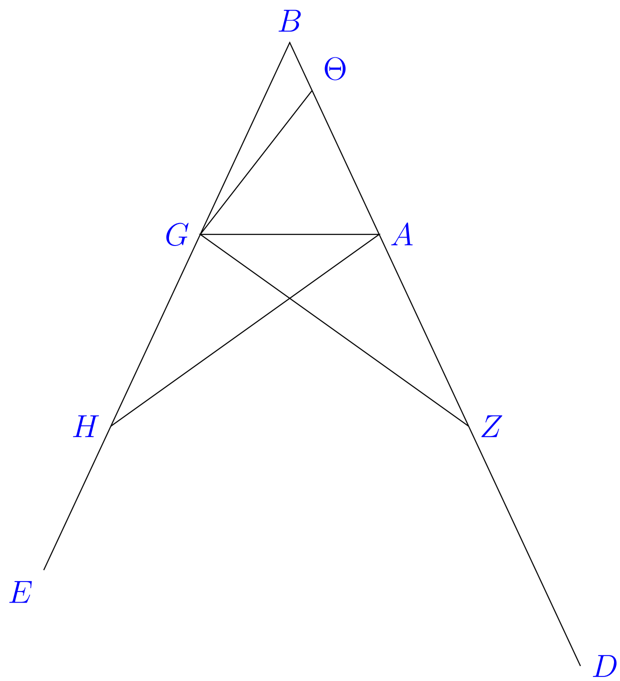

Figure 23: Proposition I.6 B, al-Nayrīzī

Proposition I.3 to proof the theorem. After concluding the proof, Tartaglia offers an additional personal reasoning, similarly based on a proof by contradiction.

We translated Tartaglia's note into English. Our notes are between square brackets.

> Note that angle *dcb* should be equal to angle *b* [premise of the proof by contradiction], but since the angle *acb* is also equal to angle *b* [premise of the theorem] it follows from the common assumptions [transitive property] that angle *dcb* must be equal to angle *acb* which we assumed to be part of, which is impossible.

Usiskin et al. [28] name this the *Isosceles Triangle Base Angles Converse Theorem*. The proof uses the definition of angle bisectors and the Congruence Theorem. The illustration consists of two triangles where the second one shows the a bisecting ray (Figure

[28]Usiskin et al., 1998, 380

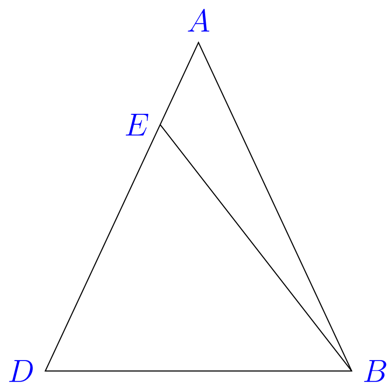

Figure 24: Proposition I.6 A, Albertus Magnus

> If two angles of a triangle are congruent, then the sides opposite them are congruent.

The original illustration shows the bisected triangle below the simple triangle. Again, we notice the preference of the authors for a simpler proof that uses concepts of symmetry.

Table

Edition	S	A	L	D
Euclid	4	0	4	1
al-Nayrīzī	10	0	12	2
Albertus	10	0	12	2
Campanus	4	0	4	1
Gerard	4	0	4	1
Tartaglia	4	0	4	1
IM	7	6	7	2

Table 5: Components of the illustrations, I.6

Proposition I.29

The illustration from *The Elements* is shown in Figure

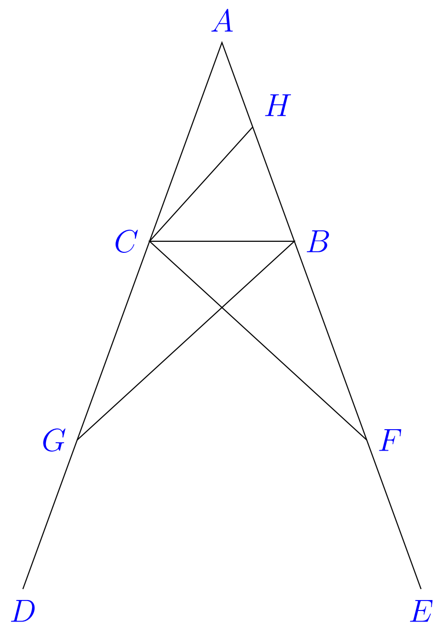

Figure 25: Proposition I.6 B, Albertus Magnus

> A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles (Heath, 2002, p. 22).

Euclid used a proof by contradiction that required Proposition I.13 and Postulate 5.

The illustration drawn by al-Nayrīzī is shown in Figure

The proof of al-Nayrīzī follows closely the one by Euclid, using an argument from contradiction and Proposition I.13 and Postulate 5 (Lo Bello, 2003a, p. 167). He also mentions a certain Agapius (Aghanis), an Athenian philosopher from the 5^{th} century. Then he offers an alternate proof that uses I.15.

Albertus Magnus is explicit in stating that he is showing Euclid's proof [29]. Then he presents the three parts of Euclid's proof. However, LoBello [30] notes that this is actually not the case.

[29] Lo Bello, 2003b, 95

[30] Lo Bello, 2003b, 193

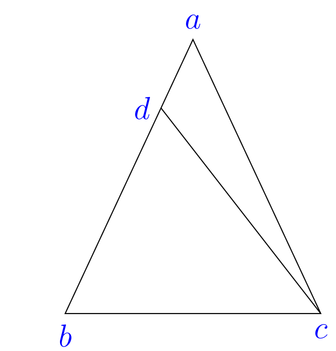

Figure 26: Proposition I.6, Campanus of Novara

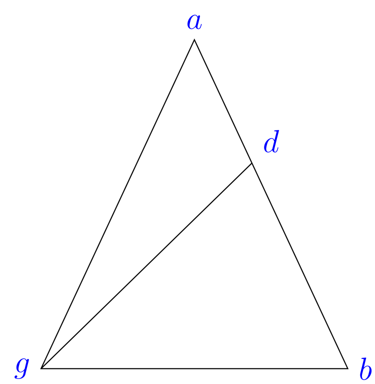

Figure 27: Proposition I.6, Gerard of Cremona

Then Albertus adds a fairly long digression (pp. 95-103) on the parallels postulate that is marred with problems caused by an error in the Arabic to Latin translation by Gerard of Cremona. Thus, we will not discuss it.

His illustration is unique among the early versions of *The Elements*. Only Albertus Magnus and Tartaglia have a "forward" slanting EF segment (see Figures

Campanus of Novara notes that this proposition is the converse of the two preceding ones (I.27 & I.28) by writing *Hec est conversa duarum precedentium*. The illustration by Campanus of Novara is slightly more complicated (Figure
Point k is the where the two lines would meet if the contradiction hypothesis were true.

The illustration by Gerard of Cremona is given in figure

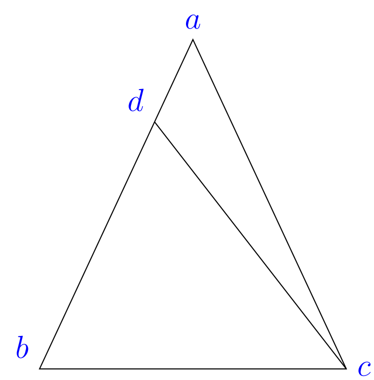

Figure 28: Proposition I.6, Tartaglia

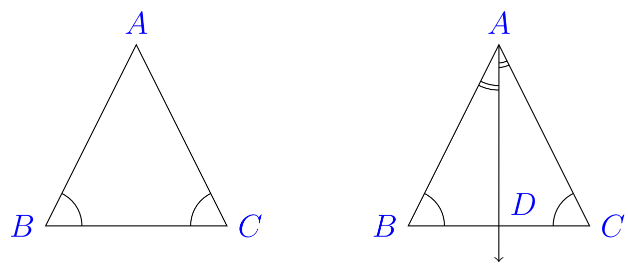

Figure 29: Proposition I.6, Integrated Mathematics

Tartaglia gives Proposition I.29 the name Theorem 20, "Theorema .20." (p. 44) and illustrates it as per Figure

Strangely, the directions of the segments *ab* and *cd* is right-to-left. We would need to look at the Zamberti edition to try to understand why. Tartaglia proves the theorem the usual way using Proposition I.13, Postulate 4, and Definition 22.

The Integrated Mathematics textbook, *Geometry* (Usiskin et al., 1998, p. 265), names this the *Alternate Interior Angles Theorem* (AIA) and states it as:

> If two parallel lines are cut by a transverse, then alternate interior angles are congruent.

Geometry gives a proof that is based on the Vertical Angles Theorem (p. 141), the Transitive Property of Congruence (p. 251), and the Parallel Lines Postulate (p. 156). The illustration is basically identical to *The Elements* with the addition that the angles are numbered for clarity (Figure

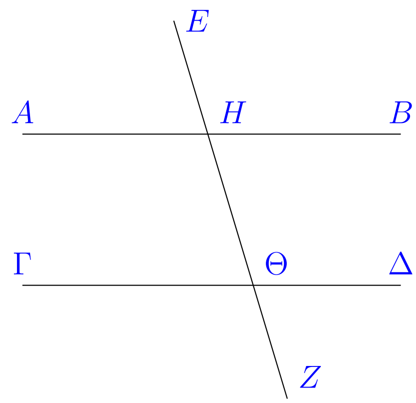

Figure 30: Proposition I.29, Euclid

Table

Edition	S	A	L	D
Euclid	3	0	8	1
al-Nayrīzī	3	0	8	1
Albertus	3	0	8	1
Campanus	5	0	9	1
Gerard	3	0	8	1
Tartaglia	5	0	9	1
IM	3	0	11	1

Table 6: Components of the illustrations, I.29

Proposition I.47

The illustration of *The Elements* is shown in Figure

> In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right triangle (Heath, 2002, p. 35).

This proposition is the famous Pythagorean Theorem. The original proof uses a triangle and rectangle congruence and addition procedure.

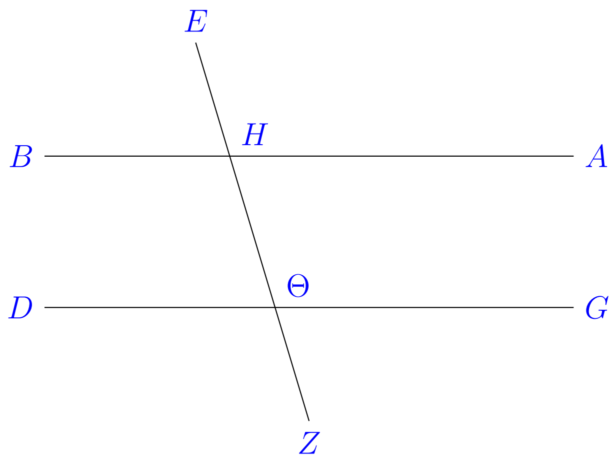

Figure 31: Proposition I.29, al-Nayrīzī

The illustration by al-Nayrīzī (Lo Bello, 2003a, p. 191) is very similar to the one by Euclid with the following differences: lines AE and KB are missing and the triangle is isosceles as we shall see below, making the squares on the legs equal, unlike the Greek drawing where they are clearly different (Fitzpatrick, 2007, p. 47). As usual the letters marking the vertexes are all switched (Figure

To proof the theorem al-Nayrīzī constructs an isosceles right triangle instead of a scalene right triangle as Euclid does. However, the proof proceeds likewise by using I.4 and I.41.

Albertus Magnus gives the number 46 to this theorem and illustrates as shown in Figure

The numbering of the theorems is off by one starting with I.46 which Albertus numbers 46. LoBello [31] traces this back to a al-Hajjaj who must have used a manuscript lacking Euclid's I.45. The version by Adelhard similarly lacks Proposition I.45.

We can notice how the drawing is the mirror image of the Greek illustration. This is most likely due to the inversion of the Arabic script. Albertus is most likely reproducing faithfully an Arabic drawing. Interestingly, al-Nayrīzī's symmetrical illustration deviates from both the Greek and al-Hajjaj. LoBello [32] noted that "the Commentary of al-Nayrizi has a complicated history." Furthermore, line AD is

[31] Lo Bello, 2003b, 214

[32] Lo Bello, 2003b, 268

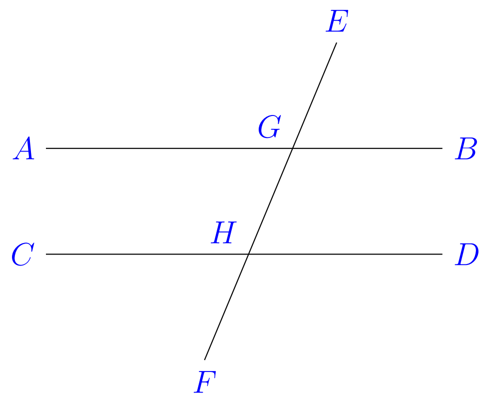

Figure 32: Proposition I.29, Albertus Magnus

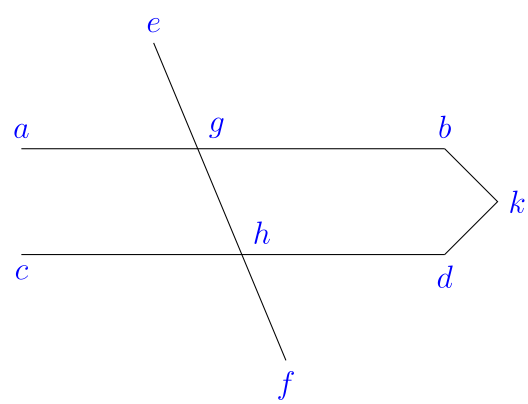

Figure 33: Proposition I.29, Campanus of Novara

missing even though the text refers to triangle ABD.

The illustration drawn by Campanus of Novara is shown in Figure

Gerard of Cremona illustrated I.47 as shown in Figure

> Angulus quoque *hbg* angulo *abd* equalis existit. Basis igitur *hg* basi *ad* equalis invenitur et triangulus *hbg* triangulo *abd* est equalis.

However, the triangles *hbg* and *abd* are not drawn. Even more strange is that the text of the proof only gives the details of using the square built on *ab*. This incongruity's would certainly have puzzled geometry student.

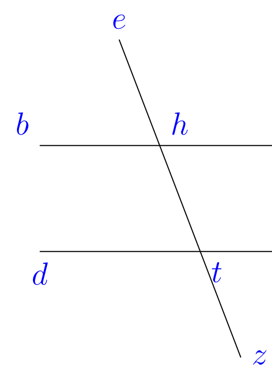

Figure 34: Proposition I.29, Gerard of Cremona

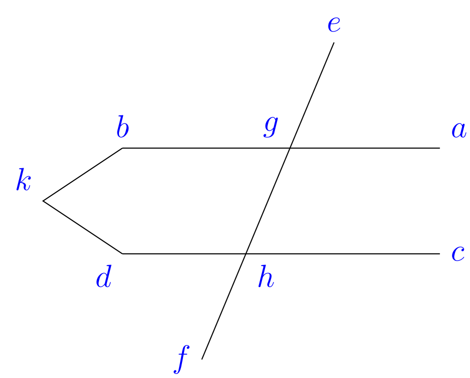

Figure 35: Proposition I.29, Tartaglia

The labels of the vertexes match those of al-Nayrīzī. The proof of the theorem is the usual triangle and rectangle congruities and additions.

Tartaglia names this Theorem 33, "Theorema .33." (p. 59). His illustration for this theorem is show in Figure

The Integrated Mathematics textbook (Usiskin et al., 1998, p. 467) base the proof on a class activity (p. 465) that prepares the students for a popular graphical/algebraic proof. Because Euclid uses a much more cumbersome triangle and rectangle congruence and addition proof (Fitzpatrick, 2007, p. 47), this approach is understandable. Euclid's text has but a single graphic. However, the high school text has three, a simple right triangle, the illustration of the statement of the theorem, and the illustration of the graphical part of the proof.

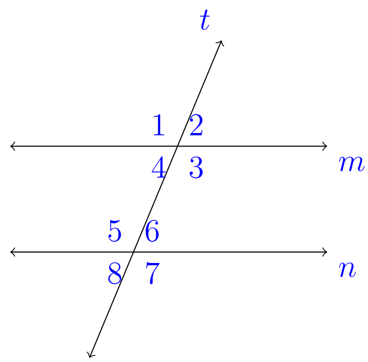

Figure 36: Proposition I.29, Integrated Mathematics

Table

Edition	S	A	L	D
Euclid	17	0	10	1
al-Nayrīzī	15	0	12	1
Albertus	16	0	11	1
Campanus	17	0	11	1
Gerard	15	0	11	1
Tartaglia	17	0	11	1
IM	12	8	8	1

Table 7: Components of the illustrations, I.47

Comparison of Illustrations

As previously noted, to obtain an idea of the complexity of the illustrations we counted the segments, arcs and number of drawings that were used to illustrate the theorems.

Generally the illustrations of Tartaglia's version are more complex than the other ones. We also note that the Integrated Mathematics textbook has generally simple illustrations. That is due to their independence from Euclid's method of proof. Usually IM uses a different type of proof that is less convoluted that the previous one. Moreover, only IM uses arcs to mark angles and uses arrow tips for lines.

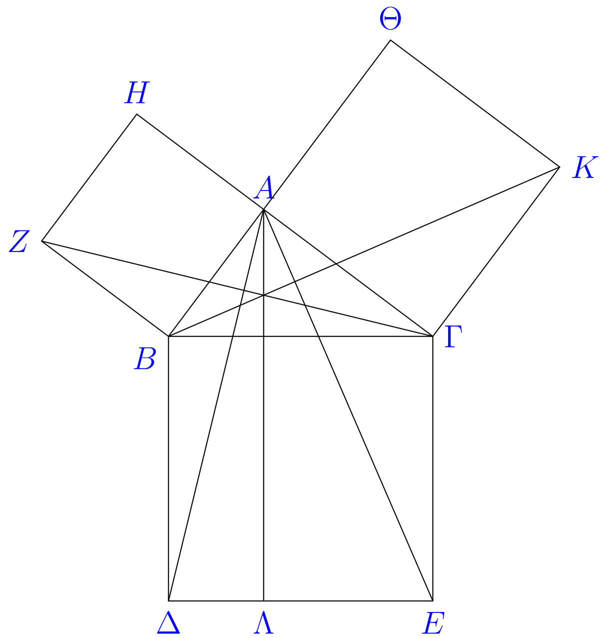

Figure 37: Proposition I.47, Euclid

4 Discussion and Conclusions

We are only aware of research on the illustrations of *The Elements* that was performed for historical purposes. Brentjes [33] analyzed the illustrations of Propositions I.9 (p. 125), I.10 (p. 126), I.11 (p. 130), I.12 (p. 130), I.13 (p. 131), to study the genealogy of a group of related Greek, Latin and Arabic manuscripts.

LoBello [34] explained why the study of the illustrations of *The Elements* is important:

> The translator, the editor, and the commentator displayed for the whole world to see whether he understood the text before him, or whether it was all Greek to him, by the accuracy, or the deficiency, of the pictures he drew, or transmitted, or corrected, or ruined.

The author also wrote (p. 268) that we can deduce the understanding of geometry of those who translated and commented *The Elements* by "what they added to the

[33] Brentjes, 1996
[34] Lo Bello, 2003b, 236

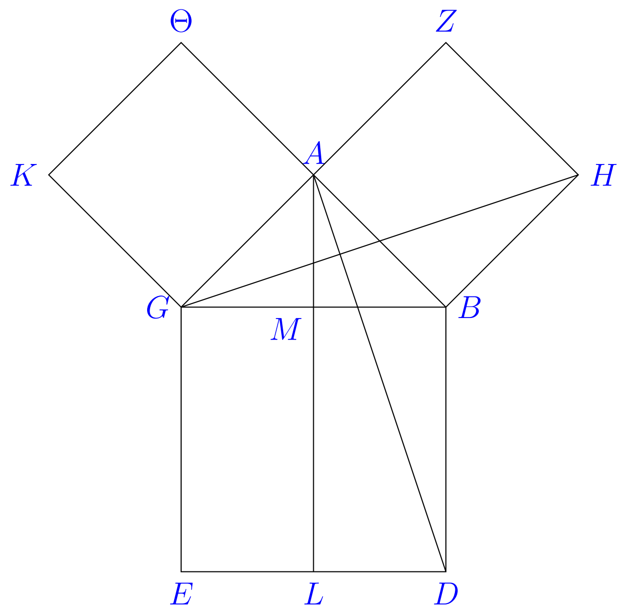

Figure 38: Proposition I.47, al-Nayrīzī

text, and by what they subtracted from it.” We believe that the same can be stated about how they illustrated their books.

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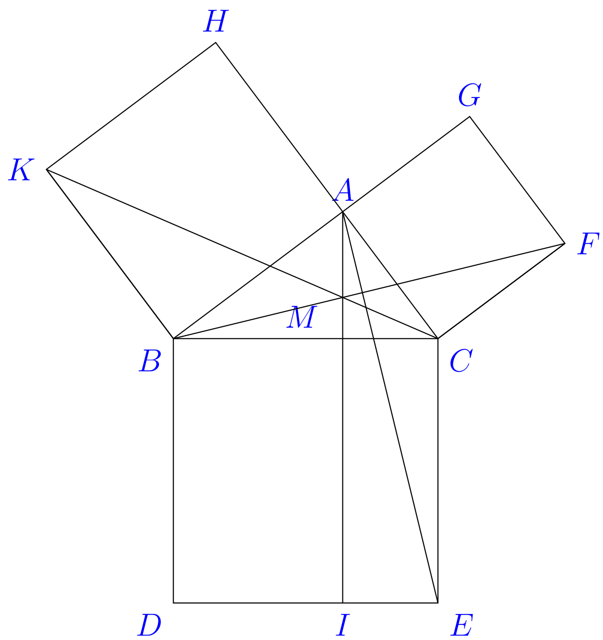

Figure 39: Proposition I.47, Albertus Magnus

Elements commonly ascribed to Gerard of Cremona. Leiden, The Netherlands: E.J. Brill.

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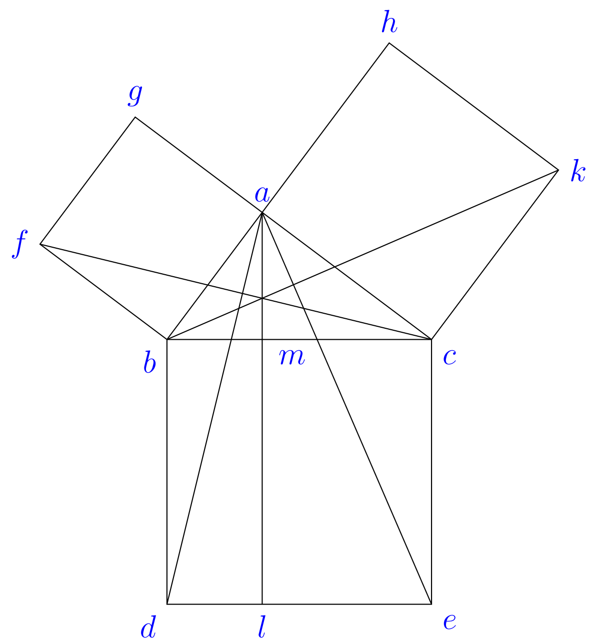

Figure 40: Proposition I.47, Campanus of Novara

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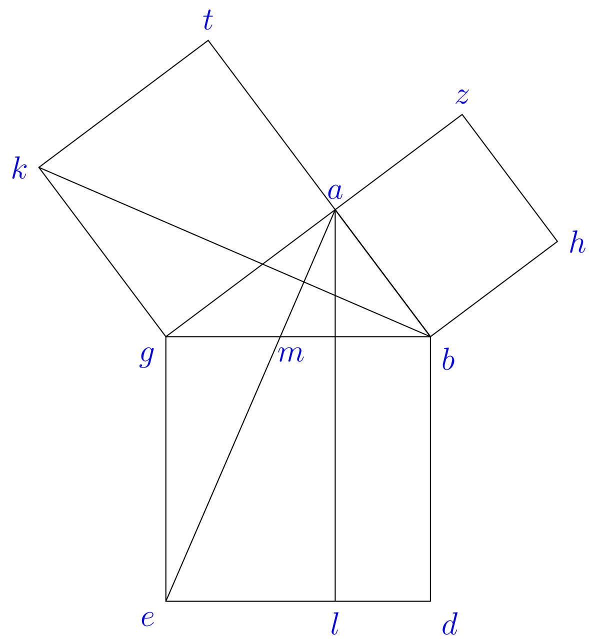

Figure 41: Proposition I.47, Gerard of Cremona

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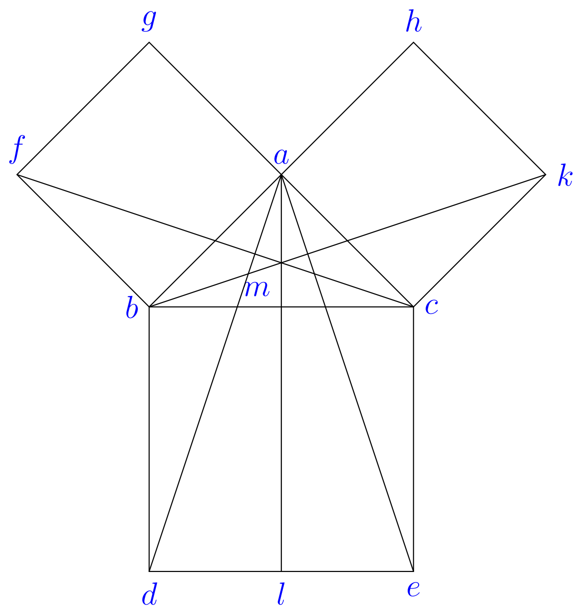

Figure 42: Proposition I.47, Tartaglia

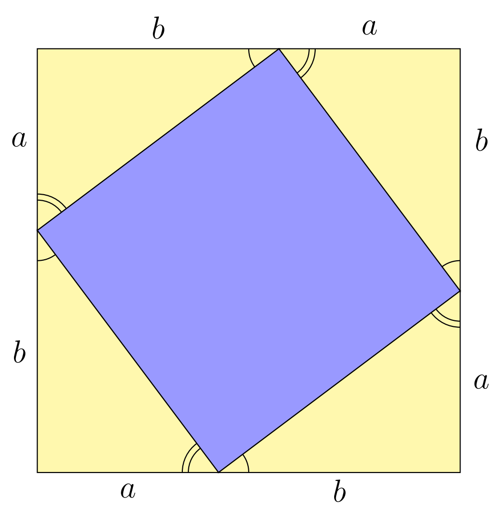

Figure 43: Proposition I.47, Integrated Mathematics