# 1 The Scandal of the Irrational

The story begins with a secret and a scandal. About 2,500 years ago, in Greece, a philosopher named Pythagoras and his followers adopted the motto "All is number." The Pythagorean brotherhood discovered many important mathematical truths and explored the ways they were manifest in the world. But they also wrapped themselves in mystery, considering themselves guardians of the secrets of mathematics from the profane world. Because of their secrecy, many details of their work are lost, and even the degree to which they were indebted to prior discoveries made in Mesopotamia and Egypt remains obscure.

Those who followed looked back on the Pythagoreans as the source of mathematics. Euclid's masterful compilation, *The Elements*, written several hundred years later, includes Pythagorean discoveries along with later work, culminating in the construction of the five "Platonic solids," the only solid figures that are regular (having identical equal-sided faces): the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron (figure 1.1). The major contribution of the Pythagoreans, though, was the concept of mathematical proof, the idea that one could construct irrefutable demonstrations of theoretical propositions that would admit of no

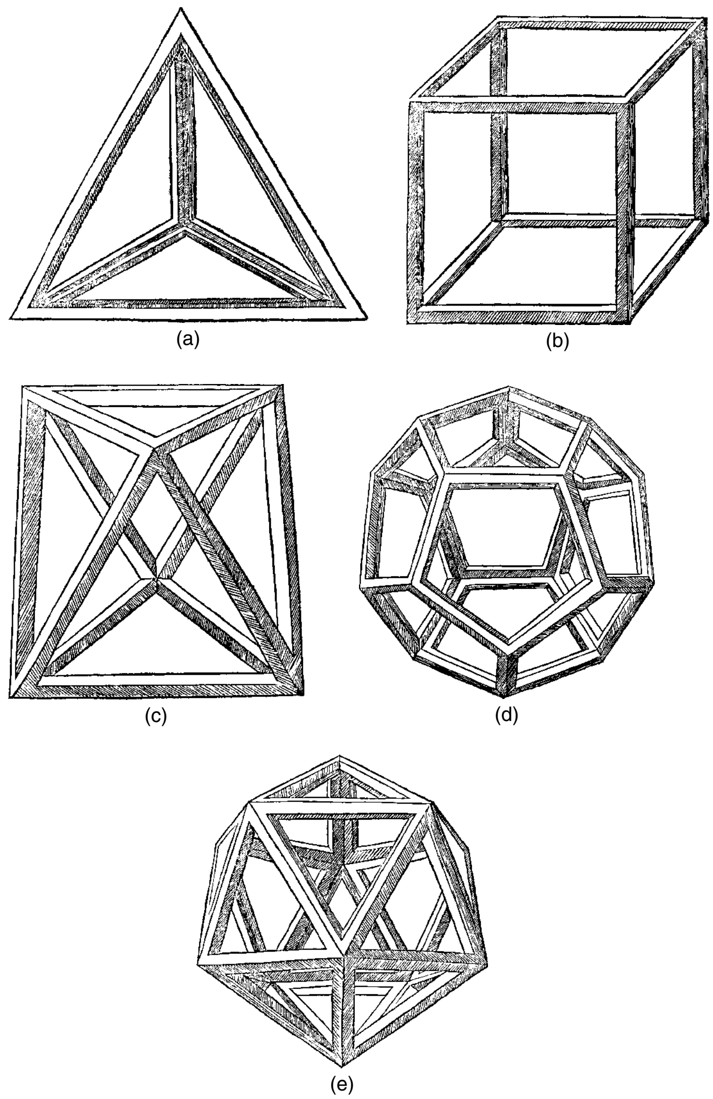


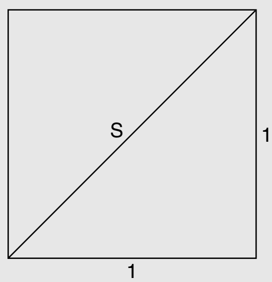

**Figure 1.1**
The five regular Platonic solids, as illustrated after Leonardo da Vinci in Luca Pacioli, *On the Divine Proportion* (1509). a. tetrahedron, b. cube, c. octahedron, d. dodecahedron, e. icosahedron.

exception. Here they went beyond the Babylonians, who, despite their many accomplishments, seem not to have been interested in proving propositions. The Pythagoreans and their followers created "mathematics" in the sense we still know it, a word whose meaning is "the things that are learned," implying certain and secure knowledge.

The myths surrounding the Pythagorean brotherhood hide exactly who made their discoveries and how. Pythagoras himself was said to have recognized the proportions of simple whole-number ratios in the musical intervals he heard resounding from the anvils of a blacksmith shop: the octave (corresponding to the ratio 2:1), the fifth (3:2), the fourth (4:3), as ratios of the weights of the blacksmith's hammers. This revealed to him that music was number made audible. (This is a good point to note an important distinction: the modern fraction $\frac{3}{2}$ denotes a breaking of the unit into parts, whereas the ancient Greeks used the ratio 3:2 to denote a relation between unbroken wholes.) Another story tells how he sacrificed a hundred oxen after discovering what we now call the Pythagorean Theorem. These stories describe events that were felt to be of such primal importance that they demanded mythic retelling.

There is a third Pythagorean myth that tells of an unforeseen catastrophe. Despite their motto that "all is number," the Pythagoreans discovered the existence of magnitudes that are radically different from ordinary numbers. For instance, consider a square with unit side. Its diagonal cannot be expressed as any integral multiple of its side, nor as any whole-number ratio based on it. That is, they are *incommensurable*. Box 1.1 describes the simple argument recounted by Aristotle to prove this. This argument is an example of a *reductio ad absurdum*: We begin by assuming hypothetically that such a ratio exists and then show that this assumption

**Box 1.1**
The diagonal of a square is incommensurable with its side




Let the square have unit side and a diagonal length $s$. Then suppose that $s$ can be expressed as a ratio of two whole numbers, $s = m{:}n$. We can assume further that $m$ and $n$ are expressed in lowest terms, that is, they have no common factors. Now note that $s^2 = m^2{:}n^2 = 2{:}1$, since the square on the hypotenuse $s$ is double the square on the side, by the Pythagorean theorem. Therefore $m^2$ is even (being two times an integer), and so too is $m$ (since the square of an even number is even). But then $n$ must be odd, since otherwise one could divide $m$ and $n$ by a factor of 2 and simplify them further. If $m$ is even, we can let $m = 2p$, where $p$ is some number. Then $m^2 = 4p^2 = 2n^2$, and $n^2 = 2p^2$. But this means that $n^2$ is even, and so too is $n$. Since a whole number cannot be both even and odd, our original assumption that $s = m{:}n$ must be wrong. Therefore the diagonal of a square cannot be expressed as a ratio of two whole numbers.

leads to an absurdity, namely that one and the same number must be both even and odd. Thus the hypothesis must have been wrong: no ratio can represent the relation of diagonal to side, which is therefore *irrational*, to use a modern term.

The original Greek term is more pungent. The word for ratio is *logos*, which means "word, reckoning, account," from a root meaning "picking up or gathering." The new magnitudes are called *alogon*, meaning "inexpressible, unsayable." Irrational magnitudes are logical consequences of geometry, but they are inexpressible in terms of ordinary numbers, and the Greeks were careful to use entirely different words to denote a number (in Greek, *arithmos*) as opposed to a magnitude (*megethos*). This distinction later became blurred, but for now it is crucial to insist on it. The word *arithmos* denotes the counting numbers beginning with two, for "the unit" or "the One" (the Greeks called it the "monad") was not a number in their judgment. The Greeks did not know the Hindu-Arabic zero and surely would not have recognized it as an *arithmos*; even now, we do not hear people counting objects as "zero, one, two, three, . . . ." Thus, the expression "there are no apples here" means "there aren't any apples here" more than "there are zero apples here."

It was only in the seventeenth century that the word "number" was extended to include not only the counting numbers from two on, but also irrational quantities. Ancient mathematicians emphasized the distinctions between different sorts of mathematical quantities. The word *arithmos* probably goes back to the Indo-European root *(a)rī*, recognizable in such English words as *rite* and *rhythm*. In Vedic India, *ṛta* meant the cosmic order, the regular course of days and seasons, whose opposite (*anṛta*) stood for untruth and sin. Thus, the Greek word for "counting number" goes back to a concept of cosmic order, mirrored in proper ritual: certain things must come *first*, others *second*, and so on. Here, due order is important; it is not possible to stick in upstart quantities like "one-half" or (worse still) "the square root of 2" between the *integers*, a word whose Latin root means unbroken or whole.

The integers are paragons of integrity; they should not be confused with magnitudes, which are divisible.

At first, the Pythagoreans supposed that all things were made of counting numbers. In the beginning, the primal One overflowed into the Two, then the Three, then the Four. The Pythagoreans considered these numbers holy, for $1 + 2 + 3 + 4 = 10$, a complete decade. They also observed that the musical consonances have ratios involving only the numbers up to four, which they called the "holy Tetractys." Out of such simple ratios, they conjectured, all the world was made. The discovery of magnitudes that cannot be expressed as whole-number ratios was therefore deeply disturbing, for it threatened the entire project of explaining nature in terms of number alone. This discovery was the darkest secret of the Pythagoreans, its disclosure their greatest scandal. The identity of the discoverer is lost, as is that of the one who disclosed it to the profane world. Some suspect them to have been one and the same person, perhaps Hippias of Mesopontum, somewhere around the end of the fifth century B.C., but probably not Pythagoras himself or his early followers. Where Pythagoras had called for animal sacrifice to celebrate his theorem, legend has it that the irrational called for a human sacrifice: the betrayer of the secret drowned at sea. Centuries later, the Alexandrian mathematician Pappus speculated that

> they intended by this, by way of a parable, first that everything in the world that is surd, or irrational, or inconceivable be veiled. Second, any soul who by error or heedlessness discovers or reveals anything of this nature in it or in this world wanders on the sea of nonidentity, immersed in the flow of becoming, in which there is no standard of regularity.

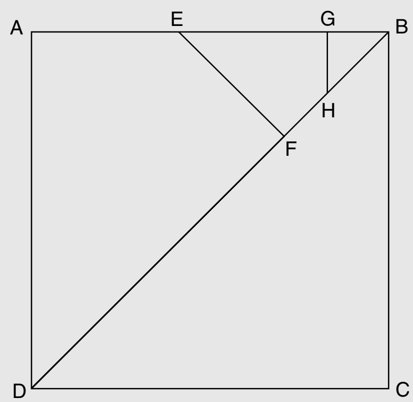
Those who immerse themselves in the irrational drown not by divine vengeance or by the hand of an outraged brotherhood but in the dark ocean of nameless magnitudes. Ironically, this is a consequence of geometry and the Pythagorean Theorem itself. When Pythagoras realized that the square on the hypotenuse was equal to the sum of the squares on the other two sides, he was very close to the further realization that, though the squares might be commensurable, the sides are not. Indeed, the argument given in box 1.1 depends crucially on the Pythagorean Theorem. That argument suggests that, had Pythagoras tried to express the ratio of the diagonal of a square to its side, he would have realized its impossibility immediately. He probably did not take this step, but his successors did.

The discovery of the irrational had profound implications. From it, Pappus drew a distinction between such "continuous quantities" and integers, which "progressing by degrees, advance by addition from that which is a minimum, and proceed indefinitely, whereas the continuous quantities begin with a definite whole and are divisible indefinitely." That is, if we start with an irreducible ratio such as 2:3, we can build a series of similar ratios in a straightforward manner: $2{:}3 = 4{:}6 = 6{:}9 = \cdots$. But if there is no smallest ratio in a series, there can be no ratio expressing the whole. Pappus's words suggest that it was this argument that may have opened the eyes of the Pythagoreans. Consider again the diagonal and side of a square. The attempt to express both of them as multiples of a common unit requires an infinite regress (box 1.2). However small we take the unit, the argument requires it to be smaller still. Again we see that no such unit can exist.

The challenge of Greek mathematics was to cope with two incommensurable mathematical worlds, arithmetic and geometry, each a perfect realm of intelligible order within itself, but with a certain tension between them. In Plato's dialogues,

**Box 1.2**

A geometric proof of the incommensurability of the diagonal of a square to its side, using an infinite regress:

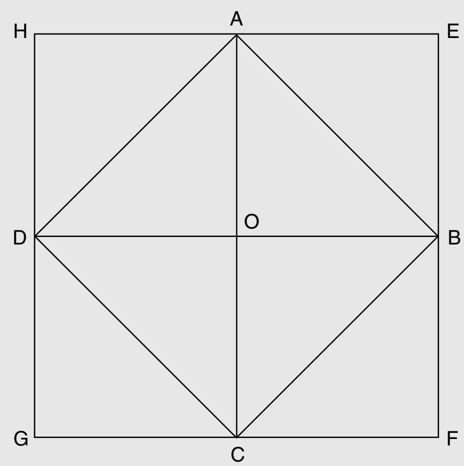



In the square *ABCD,* use a compass to lay off $DF = DA$ along the diagonal *BD*. At *F*, erect the perpendicular *EF*. Then the ratio of *BE* to *BF* (hypotenuse to side) will be the same as the ratio of *DB* to *DA*, since the triangles *BAD* and *EFB* are similar. Suppose that *AB* and *BD* were commensurable. Then there would be a segment *I* such that both *AB* and *BD* were integral multiples of *I*. Since $DF = DA$, then $BF = BD - DF$ is also a multiple of *I*. Note also that $BF = EF$, because the sides of triangle *EFB* correspond to the equal sides of triangle *BAD*. Further, $EF = AE$ because (connecting *D* and *E*) triangles *EAD* and *EFD* are congruent. Thus, $AE = BF$ is a multiple of *I*. Then $BE = BA - AE$ is also a multiple of *I*. Therefore, both the side (*BF*) and hypotenuse (*BE*) are multiples of *I*, which therefore is a common measure for the diagonal and side of the square of side *BF*. The process can now be repeated: on *EB* lay off $EG = EF$ and construct *GH* perpendicular to *BG*. The ratio of hypotenuse to side will still be the same as it was before and hence the side of the square on *BG* and its diagonal also share *I* as a common measure. Because we can keep repeating this process, we will eventually reach a square whose side is less than *I*, contradicting our initial assumption. Therefore, there is no such common measure *I*.

this challenge elicits deep responses, reaching out from mathematics to touch emotional and political life. A pivotal moment occurs in the dialogue between Socrates and Meno, a visiting Thessalian magnate who was a friend and ally of the Persian king. Meno was notoriously amoral, a greedy and cynical opportunist. Strangely, on his last day in Athens he asks Socrates over and over again whether virtue can be taught or comes naturally. Their conversation turns on the difference between true knowledge and opinion.

At the heart of their discussion, Socrates calls for a slave boy, with whom he converses about how to double the area of a square of a given size. Unlike Meno, the boy is innocent and frank; he confidently expresses his opinion that if you double the side of a square, you double its area. Their conversation is a perfect example of Socrates' practice of philosophy through dialogue. As they talk, the boy realizes that a square of double side has *four* times the area, which leaves him surprised and perplexed. The Greek word for his situation is *aporia*, which means an impasse, an internal contradiction. Just before this conversation, Socrates' questioning had revealed contradictions in Meno's confident opinions about virtue, and Meno had lashed out angrily. Socrates, he said, was like an ugly stingray that harms his victims and renders them helpless. Socrates' answer is to show how well the slave boy could take being "stung." The boy is amazed and curious, not angry. He readily follows Socrates' lead in drawing a new picture (box 1.3). In a few strokes, the real doubled square emerges by drawing the diagonals within the boy's fourfold square. Responding freely to Socrates' suggestions, the boy grasps this himself. Meno is forced to admit that the "sting" of realizing his ignorance did not harm the boy, who replaced his false opinion with a true one. The dialogue ends with Meno smoldering, foreshadowing the angry Athenians who

**Box 1.3**
Socrates' construction of the doubled square in the *Meno*




Let the original square be *AEBO*. The slave boy thought that the square on the doubled side *HE* would have twice the area, but realized that in fact that *HEFG* has four times the area of *AEBO*. At Socrates' prompting, he then draws the diagonals *AB*, *BC*, *CD*, *DA* within the square *HEFG*. Each triangle $AOB = BOC = COD = DOA$ is exactly half the area of the original square, so all four of these together give the true doubled square *ABCD*.

later voted to execute the philosopher. Their outrage points to the power of the new mathematical insights. Though Socrates had not referred to the irrationality of the diagonal, it was crucial. The process of doubling a square (an eminently rational enterprise) required recourse to the irrational, a fact that was not lost on Plato or his hearers.

Though a consequence of logical mathematics, clearly the word "irrational" already had acquired the emotional connotations it still retains. In Plato's *Republic*, Socrates jokes that young people "are as irrational as lines" and hence not yet suited to "rule in the city and be the sovereigns of the greatest things." Appropriately and yet ironically, Socrates prescribes mathematics, along with music and gymnastics, for these young irrationals to tame what is most disorderly and incommensurate in their souls. His joke points to a widely held sense that irrationality in mathematics was a troubling sign of confusion and disorder in the world, a danger as fearful as drowning. Certainly the Pythagoreans took this dire view, but Plato's dialogues open a larger perspective. What is irrational, in the soul or in mathematics, may be harmonized with the rational; to use an unforgettable image from another dialogue, the black horse of passion may be yoked to the white horse of reason.

Plato's great dialogue on the nature of knowledge rests on this mathematical crux. It is named after Theaetetus, a mathematician who is introduced to us as he is being carried back to Athens, dying from battlefield wounds and dysentery. In a flashback to Theaetetus' youth, we learn that he made profound discoveries about the irrationals and the five regular solids, and he conversed with Socrates shortly before the philosopher's trial and death. Socrates was deeply impressed with this youth, who seemed destined to do great things and who also resembled Socrates physically, down to the snub nose and bulging eyes. Also present during their conversation was Theaetetus' teacher Theodorus, an older mathematician who had proved the irrationality of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, . . ., all the way to $\sqrt{17}$, where for some reason he stopped.

Socrates' characteristic irony is not in evidence as he questions Theaetetus, who explains his discovery that there are degrees of irrationality. Though such magnitudes as the square roots of 3 or of 17 are irrational, they are still "commensurable in square," since their squares have a common measure (that is, since $(\sqrt{3})^2 = 3$ and $(\sqrt{17})^2 = 17$ are both integers). Socrates is struck with the truth and beauty of these insights and uses them as examples that lead to a broader conversation about the nature of knowledge. He reminds Theaetetus and Theodorus that he has a reputation as one who "stings" by inducing perplexity and jokingly asks Theaetetus not to denounce him as an evil wizard, explaining that he is really a midwife who helps people deliver themselves of their conceptions.

Anticipating his indictment on the very next day, Socrates justifies himself not to his angry accusers but to this gentle and gifted young man, so much like himself. Far from feeling antagonistic, Theaetetus is ready to enter a searching inquiry that begins with mathematics as a touchstone of true knowledge, testing whether other knowledge comes through the senses or more mysteriously from within the soul. Though he depicts himself as sterile, barren of wisdom, Socrates helps Theaetetus bring his conception to birth and tests its health. Socrates had often made fun of his own ugly features, but he describes Theaetetus as beautiful. Theaetetus' mathematical insight is commensurate with the bravery that will allow him to fight for his city and die with exceptional honor. Such is the courage of one who could wrestle with the irrational.

During their conversation, Socrates encourages his guests to "put themselves to torture," by which he means that they should struggle fearlessly to test and refine their opinions together. In Greek, the word for "torture" can also mean the "touchstone," a mineral that is able to distinguish gold from base metal by the mark each makes on it. This extreme metaphor has overtones of the judicial torture used to coerce truth from slaves, but Socrates uses it to signify a search for truth that defies even intense pain and humiliation. Like soldiers or athletes, Socrates and Theaetetus see in suffering the path to the superlative pleasure of ultimate truth. This they have learned from mathematics, whose study often seems painful to those who do not know the pleasure of insight. It is no wonder that Plato placed over the door to his academy the admonition: "Let no one ignorant of geometry enter here."

The discoveries of Theaetetus and the test of mathematical proof were enshrined in Euclid's *Elements*, which remains even today a living fountainhead of mathematics, invaluable for beginners as well as experienced mathematicians. Beyond presenting his own results, Euclid set the discoveries of others in order as a touchstone of mathematical lucidity and logical force. In the case of the irrational, Euclid drew on a compromise introduced by Eudoxus, who kept numbers and irrational magnitudes strictly apart, but yet in proportion. For instance, Euclid considers two numbers in a certain ratio (say 2:3) and shows how this proportion could be equal to that between two irrational magnitudes (as $2\sqrt{2}:3\sqrt{2}$ is equal to 2:3). Nevertheless, he would never mix the two distinct types so as to allow a ratio between a number and a magnitude. This was not simply mathematical apartheid but a decision to consider numbers and magnitudes as entirely distinct genera, whose mixing might lead to incalculable confusion.

Euclid's contribution went far beyond this separation of realms. In Book V, he introduces a far-reaching definition of equality or inequality that extends to ratios of irrational magnitudes. Following Eudoxus, he proposes that if we want to check whether two ratios are equal, we should multiply the terms by various integers to check whether these multiples are respectively greater, equal, or less (box 1.4). This

**Box 1.4**
Euclid's definition of equal ratio, which is applicable to any magnitude (Book V, Definition 5)

The ratio $a{:}b$ is said to be equal to the ratio $c{:}d$ if, for any whole numbers $m$ and $n$, when $ma$ is compared with $nb$ and $mc$ is compared with $nd$, the following holds: if $ma > nb$, then $mc > nd$; if $ma = nb$, then $mc = nd$; and if $ma < nb$, then $mc < nd$.

definition of equality still depends on testing multiples of magnitudes, even though the magnitudes themselves may have no common measure. It also uses *any* multiples whatever, as if to examine all possible multiples in order to determine whether the multiplied ratios could ever be equal. Thus, it is really a *test*, a trial by multiplication, a way to navigate the irrational sea. Euclid puts it in strong contrast with the way he treats whole numbers in Books VI and VII, for integers are commensurable because they have a common unit.

Euclid's most daring inquiry into the irrational occurs in Book X, which asks: Do the irrational magnitudes have some intelligible order? Can one classify them into clear categories by genus and species? He begins by showing that any magnitude can be indefinitely divided. Though implicit in geometry, he brings into prominence what later came to be called the *continuum,* meaning a continuously and endlessly divisible magnitude, as opposed to the indivisible One, whose integral multiples constitute all the counting numbers. To show this indefinite divisibility, Euclid demonstrates how we can successively subtract from any magnitude half or more of that magnitude, and then keep repeating this process until

**Box 1.5**
Euclid's statement of the indefinite divisibility of any magnitude (Book X, Proposition 1)

Take half (or more) of the given magnitude, and then the same proportion of what remains, and the same proportion yet again of what remains, continuing the process as far as necessary so that the remainder can be made less than any given line.

finally we have left a magnitude that is smaller than any given amount (box 1.5). Thus, there is no smallest magnitude, no geometrical "atom" or least possible magnitude making up all others, for if there were, all magnitudes would share that smallest magnitude as their common measure. Here again, Euclid sets in play a process indefinitely repeated, not picturable in a single figure but intelligible and logically compelling, nonetheless.

Then Euclid sets out to classify different kinds of irrationals, naming them and showing their interrelations. As Theaetetus had shown, irrationality is a relative term. The diagonal of a square is irrational compared to the side, but it can be commensurable with another line, which might be the side or diagonal of another square. What is speakable depends above all on the *relation* between figures. Euclid's classification of irrationals is intricate, though it does not go beyond what we would call the square root of the sum or difference of two square roots. He identifies such quantities in the division of a geometrical line, but we can also divide a string to make it sound different intervals. This means we can formulate a musical version of the mathematical crisis of the irrational. If we try to divide an octave (whose ratio

**Box 1.6**
The sound of square roots

Take two strings, one sounding an octave higher than the other, so that their lengths are in the ratio 2:1. Then find the geometric ratio (also called the mean proportional) between these strings, the length $x$ at which $2{:}x$ is the same proportion as $x{:}1$. This means that $2{:}x = x{:}1$; cross-multiplying this gives $x^2 = 2$. Thus, the "ratio" needed is $\sqrt{2}{:}1 \approx 1.414$, in modern decimals. This is close to the dissonant interval called the tritone, which later was called the "devil in music," namely the interval composed of three equal whole steps each of ratio 9:8. The tritone is thus $9{:}8 \times 9{:}8 \times 9{:}8 = 9^3{:}8^3 = 729{:}512 \approx 1.424$.

is 2:1) exactly at the point of the geometric mean, we get the mongrel "ratio" $\sqrt{2}{:}1$ (box 1.6). This is very close to the highly dissonant interval later called the "devil in music," the tritone. If the whole universe is based on number, such harmonic problems are critical.

Euclid presents his classification of irrationals through a hundred careful propositions. After this tour de force, he says something amazing in the final proposition of the book: From the lines already drawn, one can go on to define still other irrational lines that are "infinite in number, and none of them is the same as any of the preceding." Although his tone is impassive, this is a portentous statement. The realm of the irrational is infinite not just because there are an unlimited number of irrational magnitudes of each type but even more because there is an infinite variety of *kinds* of such magnitudes, each a different species with infinitely many examples. The discovery of the irrational disclosed an infinitely branching path.

Euclid's impassive tone does not disclose what he thought of this situation. By this final proposition, Euclid could have meant to indicate a disturbing glance into the irrational abyss, as if to say: Here lies an unfathomable, trackless sea of endlessly different magnitudes, from which one should turn away in horror. But there is another possible reading of his silence. He might have meant: Here lies an inexhaustible store of treasures, infinite in number though each is finite in magnitude. Behold, and wonder.