SUPPLEMENTARY NOTES: THE WEIERSTRASS M-TEST AND POWER SERIES

Recall that the **Taylor series** at $x = a$ of a function $f(x)$ is the series of polynomials

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

where $f^{(k)}(a)$ is the k^{th} derivative of $f(x)$[1] at $x = a$. The partial sums of the Taylor series are the **Taylor polynomials** of $f(x)$ at $x = a$.

In this note, we study the convergence of series of this form:

Definition 1. *A* ***power series*** *is a series of polynomials of the form*[2]

$$\sum_{k=0}^{\infty} c_k (x-a)^k.$$

The series is ***centered at*** $x = a$.

We will for the most part focus on series centered at $x = 0$

$$\sum_{k=0}^{\infty} c_k x^k;$$

at the end we will use the substitution $x \mapsto (x - a)$ to obtain related results for power series centered elsewhere.

Our goal is to prove and explain the following picture:

- The set of points where the series $\sum_{k=0}^{\infty} c_k x^k$ converges is an interval I, called the **interval of convergence** of the series;
- I consists of the singleton $\{0\} = [0, 0]$, the whole line $(-\infty, \infty)$, or an interval (open, closed, or half-open) whose endpoints are $\pm R$, where $0 < R < \infty$; we call R $(0 \le R \le \infty)$ the **radius of convergence** of the series
- the convergence of the series is absolute at any point interior to the interval of convergence, and uniform on any sequentially compact interval contained in I.

[1] $f^{(0)}(a)$ is the "undifferentiated " function $f(a)$ and $k!$ is the factorial of k, with $0! = 1$

[2] We have rendered the starting index $k = 0$ to underline that a power series can have a "constant" term, and it is convenient to have the index run over the non-negative integers, despite Fitzpatrick's convention of always using the natural numbers to index a sequence.

1. The Weierstrass M-test

The following analogue of the Comparison Test for numerical series is a very useful tool for proving the uniform convergence of a series of functions.

Theorem 2 (Weierstrass M-test)**.** *Suppose a sequence of functions* $f_k\colon D \to \mathbb{R}$, $k = 0, 1, \ldots,$ *satisfies the estimates*

$$|f_k(x)| \le M_k, \ k = 0, 1, \ldots, \ \textit{for all } x \in D$$

where the constants $M_k > 0$ *satisfy*

$$\sum_{k=0}^{\infty} M_k < \infty.$$

Then the series

$$\sum_{k=0}^{\infty} f_k(x)$$

converges uniformly on D*.*

Proof. Since the sum $\sum_{k=0}^{\infty} M_k$ converges, its partial sums $S_N = \sum_{k=0}^{N} M_k$ form a Cauchy sequence:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \ . \ni . \ m, n \ge N \Rightarrow |S_m - S_n| < \varepsilon.$$

For $N \le n < m$, $|S_m - S_n| = \sum_{k=n+1}^{m} M_k$; the partial sums of the series $\sum_{k=0}^{\infty} f_k(x)$ satisfy, for $N \le n < m$,

$$\left| \sum_{k=0}^{m} f_k(x) - \sum_{k=0}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \le \sum_{k=n+1}^{m} |f_n(x)| \le \sum_{k=n+1}^{m} M_k < \varepsilon,$$

showing that the sequence of partial sums of $\sum_{k=0}^{\infty} f_k(x)$ is uniformly Cauchy on D, and hence the series is uniformly convergent on D. □

2. Power Series

2.1. Interval of Convergence.

Theorem 3. *The power series* $\sum_{k=0}^{\infty} c_k x^k$ *always converges at* $x = 0$*; if it converges at* $x = b \ne 0$*, then the series converges absolutely at any point* x *in the open interval* $(-|b|, |b|)$*; furthermore, for any* r *such that* $0 < r < |b|$*, the series converges uniformly on the closed interval* $[-r, r]$*.*

Proof. That the series converges at $x = 0$ follows from the fact that every term beyond $c_0 x^0 = c_0$ is zero.

If $\sum_{k=0}^{\infty} c_k b^k$ converges, then by the Divergence Test the sequence $\{c_k b^k\}$ converges (to 0) and hence is bounded: let C be an upper bound for the absolute values of these terms:

$$\left| c_k b^k \right| \le C \text{ for all } k = 0, 1, \cdots.$$

Suppose $|x| \le r < |b|$; then

$$\left|c_k x^k\right| = \left|(c_k b^k)\left(\frac{x}{b}\right)^k\right| \le \left|c_k b^k\right|\left(\frac{r}{|b|}\right)^k \le C\left(\frac{r}{|b|}\right)^k.$$

Setting $M_k = C\left(\frac{r}{|b|}\right)^k$ and noting that the series $\sum_{k=0}^{\infty} C\left(\frac{r}{|b|}\right)^k$ is a geometric series with ratio less than 1, it follows from the Weierstrass M-Test that $\sum_{k=0}^{\infty} c_k x^k$ converges absolutely and uniformly on $[-r, r]$. □

Remark 4. *An interval can be characterized as a set $S \subset \mathbb{R}$ with the property that if $x, y \in S$ and $x < z < y$ then $z \in S$.*

Combining Theorem 3 and Remark 4 we obtain

Corollary 5. *The convergence set of a power series centered at $x = 0$*

$$I = \left\{x \in \mathbb{R} \mid \sum_{k=0}^{\infty} c_k x^k \text{ converges}\right\}$$

is an interval (open or closed or half-open). Let $R = \sup I$. The possibilities are

- ***$R = 0$:*** *$I = \{0\} = [0, 0]$: The series converges at $x = 0$ and diverges if $x \ne 0$.*
- ***$0 < R < \infty$:*** *I is an interval with finite endpoints $\pm R$, The series converges absolutely at every point x with $|x| < r$ and diverges at every point x with $|x| > R$; convergence is uniform on $[-r, r]$ for every $r < R$.*

  Convergence at the two endpoints $x = \pm R$ is not determined: depending on the series, it can converge conditionally, converge absolutely, or diverge at $x = R$ and (independently) at $x = -R$.
- ***$R = \infty$:*** *The series converges absolutely at every $x \in \mathbb{R}$, uniformly on any bounded interval.*

The number $R = \sup I$ is called the ***radius of convergence*** *of the series $\sum_{k=0}^{\infty} c_k x^k$.*

2.2. **Finding the interval of convergence (OPTIONAL).** An application of the Ratio Test can in many cases give us a way to determine the radius of convergence of a power series.

Proposition 6. *Suppose the coefficients of the power series $\sum_{k=0}^{\infty} c_k x^k$ satisfy*

$$\left|\frac{c_{k+1}}{c_k}\right| \to \rho.$$

Then the radius of convergence of the series is $R = \frac{1}{\rho}$, where we mean $R = \frac{1}{0} = \infty$ if $\rho = 0$ and $R = \frac{1}{\infty} = 0$ if $\rho = \infty$.

Proof. The series always converges at $x = 0$, as noted earlier.

If $0 < \rho < \infty$, then for any $x \neq 0$ we apply the Ratio Test to the numerical series $\sum_{k=0}^{\infty} c_k x^k$:

$$\left|\frac{c_{k+1}x^{k+1}}{c_k x^k}\right| = \left|\frac{c_{k+1}}{c_k}\right| \left|\frac{x^{k+1}}{x^k}\right| \to \rho\,|x| = \left|\frac{x}{R}\right|$$

By the ratio test, the series converges if $\left|\frac{x}{R}\right| < 1$ (*i.e.*, $|x| < R$) and diverges if $\left|\frac{x}{R}\right| > 1$ (*i.e.*, $|x| > R$).

If $\rho = 0$, the ratio test tells us that the series converges for all $x \in \mathbb{R}$, since the ratio goes to $0 < 1$. If $\rho = \infty$, for any $x \neq 0$ the terms of the series diverge to infinity. ☐

There are several more sophisticated tests that can be used to find the radius of convergence when Proposition 6 does not apply. We state them without proof:

Root Test: If $\sqrt[k]{|c_k|} \to \rho$ then the radius of convergence is $R = \frac{1}{\rho}$ as before.

limit superior: If the ratios $\left|\frac{c_{k+1}}{c_k}\right|$ do not converge, we can replace their limit with the following:

Definition 7. *Suppose $\{r_k\}$ is a non-negative sequence. Given $K \in \mathbb{N}$, let*

$$s_K = \sup\{r_k \,|\, k \geq K\}.$$

This is a decreasing sequence, since we are taking suprema over smaller sets. Since we assume $r_k \geq 0$ for all k the sequence $\{s_K\}$ is bounded and monotone, hence converges. Define the ***limit superior*** *of $\{r_k\}$as*

$$\limsup_k r_k = \lim_K s_K.$$

Then if we define ρ to be the the limit superior instead of the limit of the ratios in the Ratio Test, we get a version which (by tweaking our arguments in the proof of Proposition 6) always yields a value for the radius of convergence: if

$$\rho = \limsup \left|\frac{c_{k+1}}{c_k}\right|, \quad R = \frac{1}{\rho}$$

then the series always converges if $|x| < R$ and diverges if $|x| > R$.

2.3. **General power series.** Our discussion has focused on power series centered at $x = 0$, $\sum_{k=0}^{\infty} c_k x^k$. To handle a power series centered at $x = a$ when $a \neq 0$, $\sum_{k=0}^{\infty} c_k (x-a)^k$, we can make the substitution $y = x - a$ to obtain a new power series, $\sum_{k=0}^{\infty} c_k y^k$, centered at $y = 0$. We leave it to you to verify that applying Corollary 5 to this new series leads to the following generalization to arbitrary power series:

Theorem 8. *The convergence set of a power series centered at $x = a$*

$$I = \left\{ x \in \mathbb{R} \,|\, \sum_{k=0}^{\infty} c_k (x-a)^k \ \textit{converges} \right\}$$

is an interval (open or closed or half-open) whose midpoint is a.

The possibilities are

- **$R = 0$:** *$I = \{a\} = [a, a]$: The series converges at $x = a$ and diverges if $x \neq a$.*
- **$0 < R < \infty$:** *I is an interval with finite endpoints $a \pm R$, The series converges absolutely at every point x with $|x - a| < r$ and diverges at every point x with $|x - a| > R$; convergence is uniform on $[a - r, a + r]$ for every $r < R$.*

  Convergence at the two endpoints, $x = a \pm R$, is not determined: depending on the series, it can converge conditionally, converge absolutely, or diverge at $x = a + R$ and (independently) at $x = a - R$.
- **$R = \infty$:** *The series converges absolutely at every $x \in \mathbb{R}$, uniformly on any bounded interval.*

The number $R = \sup I$ is called the **radius of convergence** of the series $\sum_{k=0}^{\infty} c_k(x - a)^k$. Proposition 6 and its variants apply verbatim for finding this radius.