00
1-11 001
2cm2cm5mm
0 0
-4pt4pt-4pt4pt

Zbigniew Nitecki

CŒUR d'ANALYSE

Lectures on Convergence and Continuity
with Discussions and Digressions
Version: April 29, 2023

Department of Mathematics
Tufts University

Contents

Contents **1**

I The Real Numbers **9**

1 More or Less: Inequalities **11**
1.1 The Number Line 11
1.2 Inequalities and Arithmetic 15
Exercises 16

2 Intervals **19**
2.1 Absolute Value 19
2.2 Intervals 21
Exercises 23

3 Convergence of Sequences **25**
3.1 Sequences 25
3.2 Mathematical Induction 30
3.3 Convergence 31
Exercises 37

4 Arithmetic of Limits **39**
4.1 Convergence and Arithmetic 39
4.2 Series 43
Exercises 48

5 Monotone Convergence **53**
5.1 Convergence and Inequalities 53
5.2 Monotone Sequences 55
5.3 Monotone Convergence 57

Exercises . . . 63

6 Completeness of $\mathbb{R}$ **65**
6.1 The Nested Interval Theorem . . . 65
6.2 Order Completeness . . . 66
6.3 Completeness of $\mathbb{R}$ . . . 70
6.4 Cauchy Completeness . . . 72
Exercises . . . 74

7 Sequential Compactness **75**
7.1 Subsequences . . . 75
7.2 Sequentially Compact Sets . . . 78
Exercises . . . 80

8 The Cantor Set **85**
8.1 Ternary Representation . . . 85
8.2 Definition of $\mathfrak{C}$ . . . 88
8.3 Locating Points in $\mathfrak{C}$ . . . 90
8.4 Sequential Compactness of $\mathfrak{C}$ . . . 91
8.5 The Tent Map . . . 92
Exercises . . . 95

9 Counting **99**
9.1 Cardinality . . . 99
9.2 Countable Sets . . . 101
Exercises . . . 106

10 Cantor's Diagonal Argument **109**
10.1 The Cardinality of $\mathbb{R}$ . . . 109
10.2 Higher Cardinality . . . 111
Exercises . . . 114

II Continuous Functions of a Real Variable **117**

11 Continuity **119**
11.1 Continuous Functions . . . 119
11.2 Arithmetic of Continuous Functions . . . 122
11.3 Continuity and Discontinuity at a Point . . . 125
11.4 Functions Defined in Pieces . . . 127
11.5 Limits at Infinity . . . 128

Exercises . . . 129

12 Continuity of Trigonometric Functions 133
12.1 Continuity of sine and cosine . . . 133
12.2 Continuity of other trigonometric functions . . . 135
Exercises . . . 136

13 Continuity and Operations on Functions 139
13.1 Composition of Functions . . . 139
13.2 Inverse Functions . . . 140
Exercises . . . 144

14 Continuous Functions on Intervals 147
14.1 The Intermediate Value Property . . . 147
14.2 The Extreme Value Property . . . 151
Exercises . . . 152

15 Uniform Continuity 155
15.1 Uniform Continuity . . . 155
15.2 Epsilons and Deltas . . . 157
Exercises . . . 160

III Convergence and Continuity in Higher Dimensions 163

16 $\mathbb{R}^n$ 165
16.1 Vectors . . . 165
16.2 The length of a vector . . . 166
16.3 Convergence in $\mathbb{R}^n$ . . . 170
Exercises . . . 175

17 Topology of $\mathbb{R}^n$ 177
17.1 Open Sets . . . 177
17.2 Closed sets . . . 180
17.3 The duality between open and closed sets . . . 182
17.4 Cluster points; perfect sets . . . 184
17.5 Cantor's Second Non-Countability Proof . . . 185
Exercises . . . 187

18 Continuity in $\mathbb{R}^n$ 189
18.1 Continuity of Functions on $\mathbb{R}^n$ . . . 189

18.2 Continuity and Topology . . . 191
18.3 Mappings between Euclidean spaces . . . 192
18.4 Continuity on a Subset . . . 194
Exercises . . . 195

19 Sequentially Compact Sets in $\mathbb{R}^n$ 197
19.1 Sequential Compactness in $\mathbb{R}^n$ . . . 197
19.2 The Bolzano-Weierstrass Theorem . . . 198
19.3 Sequentially Compact Sets in $\mathbb{R}^n$ . . . 200
Exercises . . . 202

20 The Finite Covering Property 205
20.1 Heine-Borel Theorem in $\mathbb{R}$ . . . 205
20.2 Open Covers and Subcovers . . . 207
20.3 The Borel-Lebesgue Theorem . . . 209
Exercises . . . 213

21 Continuous Mappings on Compact Sets 215
21.1 The Extreme Value Theorem . . . 215
21.2 Compactness and Mappings . . . 217
21.3 Uniform Continuity . . . 218
21.4 Using the Finite Covering Property . . . 218
Exercises . . . 221

22 Path-Connected Sets in $\mathbb{R}^n$ 225
22.1 Convex Sets in $\mathbb{R}^n$ . . . 225
22.2 Path-Connected Sets . . . 227
Exercises . . . 228

23 Connected Sets in $\mathbb{R}^n$ 231
23.1 Separated Sets . . . 231
23.2 Connected Sets . . . 236
23.3 Components . . . 241
Exercises . . . 244

IV Function Spaces 247

24 Sequences of Functions 249
24.1 Pointwise Convergence: Examples . . . 249
24.2 Pointwise Convergence: Features . . . 253

Exercises . . . 258

25 Uniform Convergence 261
25.1 Uniform Convergence . . . 261
25.2 The Uniform Cauchy Criterion . . . 263
25.3 Uniform Convergence and Continuity . . . 266
Exercises . . . 270

26 Series 273
26.1 Convergence Tests . . . 273
26.2 Absolute versus Conditional Convergence . . . 278
Exercises . . . 285

27 Series of Functions 289
27.1 The Weierstrass M-Test . . . 289
27.2 Power Series . . . 291
27.3 Interval of Convergence . . . 294
27.4 Abel's Continuity Theorem . . . 299
27.5 Trigonometric series (optional) . . . 303
Exercises . . . 308

28 The Space of Continuous Functions 311
28.1 The Space $\mathcal{C}([0,1])$ . . . 311
28.2 Convergence in $\mathcal{C}([0,1])$ . . . 312
28.3 Continuous Mappings on $\mathcal{C}([0,1])$ . . . 313
28.4 Open and Closed Sets in $\mathcal{C}([0,1])$ . . . 316
Exercises . . . 317

29 Normed Linear Spaces 321
29.1 Subspaces of $\mathcal{C}([0,1])$ . . . 321
29.2 Normed Linear Spaces . . . 322
Exercises . . . 326

30 Compactness in $\mathcal{C}([0,1])$ 329
30.1 The Failure of Bolzano-Weierstrass . . . 329
30.2 The Arzelà-Ascoli Theorems . . . 332
Exercises . . . 338

31 Application to Differential Equations 341
31.1 Initial-Value Problems . . . 341
31.2 Existence vs. Uniqueness of Solutions . . . 343

31.3 The Cauchy-Lipschitz Theorem . . . 345
31.4 The Cauchy-Peano Existence Theorem . . . 347
Exercises . . . 353

32 Polynomial Approximation **355**
32.1 Representation of Functions . . . 355
32.2 The Weierstrass Approximation Theorem . . . 356
32.3 The Stone-Weierstrass Theorem . . . 371
Exercises . . . 376

33 Nowhere Differentiable Functions **379**
33.1 Bolzano's Example . . . 380
33.2 Weierstrass's Example . . . 389
Exercises . . . 394

34 Space-Filing Curves **397**
34.1 The Examples of Peano and Hilbert . . . 397
34.2 Peano's Example . . . 397
34.3 Hilbert's Strategy . . . 402
34.4 Peano's Example via Hilbert's Strategy . . . 408
34.5 Lebesgue's Example . . . 416
Exercises . . . 420

V Metric Spaces **425**

35 The Notion of a Metric **427**
35.1 Definition and Examples . . . 428
35.2 Convergence and Topology in Metric Spaces . . . 436
35.3 Continuous Mappings . . . 443
Exercises . . . 449

36 Completeness in Metric Spaces **455**
36.1 Complete Metric Spaces . . . 455
36.2 Cauchy Completion . . . 459
Exercises . . . 470

37 The Contraction Mapping Theorem **475**
37.1 Contractions and Fixed Points . . . 475
37.2 Initial-Value Problems Revisited . . . 478
Exercises . . . 482

38 Compactness in Metric Spaces **485**
38.1 Sequential Compactness and Coverings 487
38.2 The Resurrection of Heine-Borel 490
38.3 The Extreme Value Property 493
38.4 Subsets vs. Subspaces 496
Exercises 498

39 Connectedness in Metric Spaces **501**
39.1 Path-Connected Sets in a Metric Space 501
39.2 (Dis)Connected Metric Spaces and Subsets 502
39.3 The Intermediate Value Property 505
Exercises 506

40 The Hausdorff Metric **515**
40.1 Definition of the Hausdorff Metric 515
40.2 Convergence in $\mathfrak{H}(X)$ 519
40.3 Hausdorff Metric in Compact Metric Spaces 523
40.4 Iterated Function Systems and Fractals 525
Exercises 527

VI Topological Spaces **531**

41 The Notion of a Topology **533**
41.1 A Motivating Example 533
41.2 Definitions of Topological Space 534
41.3 Convergence and Continuity 538
41.4 Examples of Topological Spaces 541
Exercises 547

Bibliography **553**

Index **575**

Part I

The Real Numbers

Lecture 1

More or Less: The Algebra and Geometry of Inequalities

1.1 The Number Line

"Numbers " come in several flavors, and in varying degrees of abstraction.

The **natural numbers** $1, 2, 3, \ldots$ are associated with counting–either counting how many elements belong to a set ("cardinal" numbers), or locating a position in a list ("ordinal" numbers). Addition of natural numbers is associated to counting the number of elements in the union $A \cup B$[1] of two disjoint[2] sets A and B: if A has m elements and B has n then $A \cup B$ has $m + n$. If we think of the natural numbers as a list, with the successor $m + 1$ of $m \in \mathbb{N}$ to the right of m, then adding n to m has the effect of moving n places to the right. Multiplying m by n is the same as adding together n copies of m. Addition and multiplication are related by the **distributive law**: $m(n + k) = mk + nk$. The collection of all natural numbers is denoted $\mathbb{N}$.

The **integers** $\mathbb{Z}$ consist of the natural numbers, or in this context the **positive integers** together with the number zero (0) and the **negative integers** $-1, -2, -3, \ldots$, arranged in succession to the left of $\mathbb{N}$. The identities $n + 0 = n$, $-n + n = 0$ and $-n = (-1)n$ together with the distributive law lead to a unique extension of addition and multiplication from $\mathbb{N}$ to $\mathbb{Z}$. **Subtraction** of n from m is defined as $m - n = m + (-n)$.

[1] The union $A \cup B$ consists of all elements that belong to either A or B or both.

[2] A and B are **disjoint** if they share no common elements–that is, their **intersection** contains no elements–it is "empty": $A \cap B = \emptyset$.

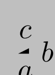
The **rational numbers**, denoted $\mathbb{Q}$, are represented by **fractions** $\frac{p}{q}$; we adopt the convention that in this notation, the **numerator** p is an *integer* but the **denominator** q is a *natural number.* To incorporate these in a geometric representation, we think of the integers $\mathbb{Z}$ as spaced one unit of length apart on a line.
Then we locate the fraction $\frac{p}{q}$ by taking the interval whose endpoints are 0 and p, dividing it into q subintervals of equal length, and then locating $\frac{p}{q}$ at the "other" endpoint of the subinterval starting at 0. This scheme puts the fraction $\frac{mp}{mq}$, where $m \in \mathbb{N}$ is any positive integer, at the same location as $\frac{p}{q}$. This location is the **ratonal number** represented by any of these fractions. Among them, there is one which minimizes the denominator; it is called a **reduced fraction** and is characterized by the fact that p and q are relatively prime. Every representative of the rational represented by a reduced fraction $\frac{p}{q}$ has the form $\frac{mp}{mq}$ for some $m \in \mathbb{N}$.

Figure 1.1: $\mathbb{Q}$, the rationals

One major difference between $\mathbb{Z}$ and $\mathbb{Q}$ is that every $x \in \mathbb{Z}$ has a "successor" $(x+1)$ and a "predecessor" $(x-1)$, while it makes no sense to talk about the "next" rational after x (or the "last" one before it), since between any two distinct rationals, say $x < y \in \mathbb{Q}$, there exists another rational $x < z < y$ (for example, their midpoint $z = \frac{1}{2}(x+y)$).

Digression:
Irrational Numbers

As every Calculus student knows, not every quantity can be expressed as a rational number. The most basic example comes from the **Pythagorean Theorem**, that *the square of the length of the hypotenuse of a right triangle equals the sum of the squares of the other two sides.*



Ironically, the historical figure after which this result is named, Pythagoras of Samos (*ca.* 580-500 BC), is also associated with a point of view that was destroyed via this equation. According to this view, any two quantities are **commensurable**, meaning that they are both multiples (via natural numbers) of some common divisor. About a century after Pythagoras' death, it was discovered by Hippasus of Metapontum (*ca.* 400 BC) (a follower of the philosophical school Pythagoras had started) that when $a = 1 = b$, then c (which we now call $\sqrt{2}$) is *incommensurable* with a.[a] The standard argument establishing this fact is a classic example of **proof by contradiction**, a method of proof which amounts to showing that the statement to be proved can't possibly be false.

Theorem 1.1. *There is no rational number whose square equals* 2.

Proof:
Suppose that $\frac{p}{q}$ is a fraction satisfying

$$\left(\frac{p}{q}\right)^2 = 2,$$

or

$$p^2 = 2q^2. \tag{1.1}$$

We can assume that p is positive and the fraction $\frac{p}{q}$ is irreducible, and in particular that p and q are *not both even*.
By Equation (**??**), p^2 is even. It follows that q itself is even, since it is easy to see that the square of an odd natural number is itself odd.
This says that

$$p = 2m \tag{1.2}$$

for some $m \in \mathbb{N}$. Substituting this in Equation (**??**) leads to

$$2m^2 = q^2 \tag{1.3}$$

so (since q^2 is even), q is *also* even.
But this contradicts our assumption that the fraction $\frac{p}{q}$ is reduced, or more precisely that p and q can't *both* be even. The contradiction proves the theorem.[b] □

[a] Various versions of this story have Hippasus either banished or thrown overboard at sea for uncovering and publicizing this inconvenient truth.

[b] In effect, we have established that there is no *reduced* fraction representing $\sqrt{2}$.

The upshot of this is that we need to go beyond $\mathbb{Q}$ to get our hands on what we mean by a "number". Essentially, all arithmetic calculation is done on rationals: an expression like $1+\sqrt{2}$ or $\sin(\pi/\sqrt{2})$ or $e^{\sqrt{2}}$ is simply an abstract manipulation of symbols unless we can ground it in some unified conception of "**real numbers**", which we think of as points on the "number line", denoted $\mathbb{R}$.

These days we are used to thinking of real numbers in terms of **decimal expansions**, which are infinite strings of digits. In practice, though, we "calculate" (whether by hand or on a computer) with truncated versions of these sequences, which in turn represent rational numbers. Our picture of real numbers as points on a "number line" is a useful source of intuition, but to understand real numbers in a careful way we need to connect this geometric intuition to operational notions, starting with (rational) arithmetic.

Such a connection was developed in the nineteenth century, starting with the formulation and proof of the Intermediate Value Theorem[**?**] by Bernhard Bolzano (1781-1848) in 1817 and two sets of lecture notes published by Augustin-Louis Cauchy (1789-1857) in 1821 and 1823: *Cours d'analyse de l'école polytechnique*(1821) [**?**, **?**] and *Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal* (1823)*Résumé des lecons sur le calcul infinitesimal* (1823)][**?**]. These works initiated a careful study of convergence, based on the arithmetic of inequalities. In the course of the nineteenth century, this point of view was developed and extended to a rigorous theory real numbers, and of real-valued functions; in addition to Bolzano and Cauchy some notable contributors were Niels Henrik Abel (1802-1829), Peter Lejeune-Dirichlet (1805-1859), Bernhard Riemann (1826-1866), Georg Ferdinand Cantor (1845-1918), and Karl Weierstrass (1815-1897). This theoretical development is sometimes referred to as the "arithmetization of analysis".

1.2 Inequalities and Arithmetic

In practice, the relative position of two rational numbers is easy to decide from a pair of representative fractions:[3] $\boxed{\frac{m}{n} < \frac{p}{q} \text{ precisely if } mq < np.}$

However, when a number, such as $\sqrt{2}$, is specified by some property, its position relative to other numbers must be deduced from this property by manipulating inequalities.

[3]We reiterate our convention that the denominator of a fraction is a natural number, so we always assume when we write $\frac{p}{q}$ that, while p can be a positive or negative integer (or zero), q is a *positive* integer.

The manipulation of *equalities* is governed by a single, simple principle:
If the same arithmetic operation is applied to two equal numbers, the results are equal.
An analogous simple rule does not hold when the same arithmetic operation is applied to two *unequal* properties, say $x < y$–this inequality does not necessarily continue to hold for the results of an arithmetic operation. Depending on the operation, the signs involved, as well as other considerations, the inequality may be preserved or reversed. An exhaustive list of rules concerning this question would probably be very complicated, but we can point to some particularly common pitfalls to avoid. What we present can be justified for *rational* numbers from our definitions, but with considerably more work they can also be extended to irrational numbers.

A surprisingly useful initial observation is
$\boxed{x < y \text{ if and only if } x - y < 0, \text{ or equivalently } y - x > 0}$.
Very often, a question about inequality is much easier to answer when it is posed as finding the *sign* of some related quantity. An equivalent formulation (based on the trivial calculation $(x + y) - y = x$) is

Remark 1.2. $\boxed{\textit{If } y > 0 \textit{ then for every } x,\ x < x + y.}$

Similarly, the fact that $x - y$ and $y - x$ have opposite signs yields

Remark 1.3. $\boxed{x < y \textit{ if and only if } -y < -x.}$

The fact that the product of two positive numbers is positive, together with Remark **??**, can be used to show

Remark 1.4. *The product of two numbers is positive if they have the same sign and negative if they have opposite sign.*

Reciprocals are a bit trickier:

Remark 1.5. *If x and y have the same sign, then $x < y$ if and only if $\frac{1}{y} < \frac{1}{x}$.*

This follows from the calculation

$$\frac{1}{x} - \frac{1}{y} = \frac{y - x}{xy}$$

together with Remark **??**. What about the other case (different sign)?

An illustration of the practical usefulness of switching from an inequality question to a sign question is

Question 1.6. *Suppose we know that $x < y$ and $u < v$; what is the relation between $x - u$ and $y - v$?*

We note that if u and v are nearly zero, then $x - u$ remains less than $y - v$, but if u and v are very high, then $x - u$ is greater than $y - v$,
To decide the threshold between the two, we ask for the sign of $(x - u) - (y - v) = (x - y) - (u - v)$, and it becomes clear that

$\boxed{x - u < y - v \text{ precisely if } (x - y) < (u - v).}$

Exercises

1. Show that if $m \in \mathbb{N}$ then m^2 is even if and only if m is even. *Note that there are two things to prove:* if m is even, then so is m^2, and if m^2 is even then m is even.

2. Each of the following statements is true whenever all the letters represent positive real numbers. For each statement, either give an example (involving some negative numbers) for which it is false, or prove that it is true for *all* real numbers:

   a) If $a < b$ then $\frac{1}{a} > \frac{1}{b}$. (b) If $a < b$ then $-a > -b$,

3. Suppose $0 < x < y$ and $0 < u < v$. What is the relation between $\frac{x}{u}$ and $\frac{y}{v}$?

4. **Cauchy's mediant lemma:** [**?**, Thm 1, p. 13][4]

   a) Show that if
   $$0 < \frac{a_1}{b_1} \leq \frac{a_2}{b_2}$$
   Then
   $$\frac{a_1}{b_1} \leq \frac{a_1 + a_2}{b_1 + b_2} \leq \frac{a_2}{b_2}.$$

[4] The fraction appearing in the middle of the inequality in (a) is called the **mediant** or **Farey sum** of the outer fractions. If we consider all the reduced proper fractions with denominator less than a given bound, listed in increasing order, then each entry is the mediant of its two immediate neighbors. This was observed (without proof) by John Farey (1766-1826)[**?**]; Cauchy then published a proof of this [**?**], crediting Farey with the idea of mediants. However, it seems this same observation had been used much earlier by Charles Haros when preparing tables giving the conversion of fractions to decimal form [**?**]. The corresponding quantity in (b) is also called the mediant of the collection of fractions a_i/b_i; Cauchy was interested in this as a kind of average.

Note that we defined a fraction to have a positve denominator (but a numerator of either sign).

b) Extend this (by induction on n) to a collection of n positive fractions $\frac{a_i}{b_i}$, $i = 1, \ldots, n$:

$$\min\left\{\frac{a_i}{b_i}\right\} \leq \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \max\left\{\frac{a_i}{b_i}\right\}$$

Is the assumption that both fractions are positive necessary for this to hold?

Lecture 2

Intervals

2.1 Absolute Value

The fact that negation reverses inequalities leads us to the following

Definition 2.1. *The **absolute value** of any number x is defined to be*

$$|x| = \max\{x, -x\}.$$

That is, the absolute value of any number is either the number itself or its negative, whichever is higher:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

Thus, the absolute value of x is a non-negative number, which expresses the length of the interval with endpoints 0 and x. This can also be interpreted as the **size** of the number x. We shall take some care to distinguish saying that x is *less* than y–or y is *higher* than x ($x < y$)–from saying that x is *smaller* than y–or y is *larger* than x ($|x| < |y|$).
For many reasons (not least of which is that inequalities among positive numbers are simpler than inequalities among numbers that can sometimes be negative) we will very often be trying to establish inequalities among *absolute values* rather than numbers in general. To this end, it is useful to note the following

Remark 2.2. *Every number, as well as its negative, is less than or equal to its absolute value:*

$$x \leq |x| \text{ and } -x \leq |x|,$$

and for any other number y,

$|x| \leq y$ if and only if both $x \leq y$ and $-x \leq y$.

(After all, the *higher* of two numbers is less than y precisely if each of the two numbers individually is less than y.)
We shall make frequent use of three basic properties of the absolute value, codified in the following

Proposition 2.3. *The absolute value function satisfies:*

Positive-Definite: *For all x, $|x| \geq 0$, and $|x| = 0$ precisely when $x = 0$.*

Scaling: *For any x and y, $|xy| = |x|\,|y|$.*

Triangle Inequality: *For any x and y, $|x + y| \leq |x| + |y|$.*

The first property is an immediate consequences of the definition, but the other two require proof.

1. **Proof of scaling:**

   Case 1 If x and y have the same sign, then xy is non-negative, and hence equals $|xy|$. In this case, if x and y are both positive, then also $xy = |x|\,|y|$, while if both are negative, then $|x|\,|y| = (-x)(-y) = xy = |xy|$

   Case 2 If they have opposite signs, then xy is negative, so $|xy| = -xy$. Now, either $|x| = -x$ or $|y| = -y$, *but not both*, so $|x|\,|y| = -xy$ as well.

   □

2. **Proof of triangle inequality:**

   We need to show that for any two numbers x and y,

   $$\max\{(x + y), -(x + y)\} \leq |x| + |y|\,.$$

   Applying Remark **??** to x and y individually, we have the four inequalities

   $$\begin{aligned} x \leq |x| &\text{ and } -x \leq |x| \\ y \leq |y| &\text{ and } -y \leq |y|\,; \end{aligned}$$

adding the first and third inequality (and using the basic fact about adding unequals) yields

$$x + y \leq |x| + |y|$$

while adding the second and fourth inequality yields

$$-x - y \leq |x| + |y|$$

or $-(x + y) \leq |x| + |y|$. These two observations let us apply Remark **??** again, this time to $(x + y)$, to get

$$|x + y| \leq |x| + |y|$$

which is the triangle inequality. □

The absolute value $|x|$ of a number can be interpreted as the distance from x to the origin, 0. More generally, we can measure the distance between two distinct numbers, x and y, as

$$\text{dist}(x, y) = |x - y|$$

From this point of view, the triangle inequality for the absolute value translates to the statement

Triangle Inequality for Distance:

For any three points x, y, z, $\text{dist}(x, y) \leq \text{dist}(x, z) + \text{dist}(z, y)$.

This is the same as the triangle inequality for absolute value, applied to

$$|x - y| = |(x - z) + (z - y)|\,.$$

Geometrically, if we draw the collapsed triangle whose vertices are the three points, it is the statement that each side of the triangle is no longer than the sum of the other two.

2.2 Intervals

While subsets of $\mathbb{R}$ come in many different varieties, one variety of subset which occurs very frequently (for example, as the domain of a function) is the *interval*:

Definition 2.4. *An **interval** is a set I of real numbers with the property that for any two elements $x, y \in I$, every point between x and y is also an element of I.*

We have a special notation for intervals, which distinguishes several subvarieties:

- The **closed interval** with endpoints $a \leq b$ is

$$\boldsymbol{[a,b]} = \{x \in \mathbb{R} \mid a \leq x \text{ and } x \leq b\}.$$

- The **open interval** with endpoints $a \leq b$ is

$$\boldsymbol{(a,b)} = \{x \in \mathbb{R} \mid a < x \text{ and } x < b\}.$$

- We can also define **half-open** intervals:

$$\boldsymbol{(a,b]} = \{x \in \mathbb{R} \mid a < x \text{ and } x \leq b\}$$
$$\boldsymbol{[a,b)} = \{x \in \mathbb{R} \mid a \leq x \text{ and } x < b\}.$$

All of these sets consist of the points "between a and b" ; the distinction between them hinges on whether a (*resp.* b) is included in, or excluded from, the set. This distinction may seem a bit esoteric, but as we shall see it can often play a crucial role in our study of real numbers. An interval specified in any of these ways as the collection of points between two specified real numbers $a, b \in \mathbb{R}$ is called a **bounded interval** and the points a and b are its **endpoints**. When sketching intervals, we mark an *included* endpoint by a filled-in dot, and an *excluded* one by a "hollow" dot:

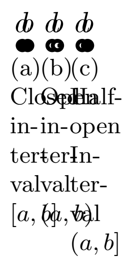


Figure 2.1: Bounded Intervals

Definition **??** encompasses, in addition to intervals defined by two inequalities, intervals defined by a single inequality, like the set of strictly positive numbers noted before Definition **??**. We adapt our notation to these sets by use of the symbols ∞ and $-\infty$, which we can think of as the right and left "ends" of the number line, so *every* real number x satisfies the two inequalities $-\infty < x$ and $x < \infty$. This device lets us write $(0, \infty)$ for the set of strictly positive numbers, and $(-\infty, 0]$ for the set of non-positive numbers–we can even write

$$\mathbb{R} = (-\infty, \infty)$$

for the interval defined by no inequalities at all. It is important to remember that $\pm\infty$ *are not numbers*–we cannot perform arithmetic operations on them in a meaningful way–and by convention we never write them as "included" endpoints of an interval. We refer to an interval with at least one "infinite endpoint" as an **unbounded interval**.

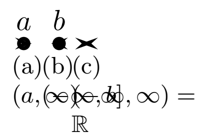


Figure 2.2: Unbounded Intervals

Exercises

1. Each of the following statements is true whenever all the letters represent positive real numbers. For each statement, either give an example (involving some negative numbers) for which it is false, or prove that it is true for *all* real numbers:

   a) $|a - b| \leq |a| + |b|$. (b) $|a - b| \geq |a| - |b|$.

2. What is closer to 1: $\frac{3}{4}$ or $\frac{4}{3}$? As a general rule, which positive real numbers are closer to 1 than their reciprocal? Justify your answer.

3. If $p(x)$ is a polynomial of degree n, then for any $a \in \mathbb{R}$, there are at most n real numbers that satisfy the equation $p(x) = a$. So a *solution* of such an equation is a list of *all* (real) numbers that satisfy it. In general, an *inequality* can be satisfied by infinitely many numbers, so such a list is impossible. But we could reasonably say that we have "solved" the inequality if we can express the collection of all the numbers that satisfy it as a disjoint union of intervals.

   The function
   $$f(x) = x^2 - x^4$$
   has a local maximum at $x = \pm\frac{1}{\sqrt{2}}$ and a local minimum at $x = 0$.

Solve each of the following:

a) $f(x) = -2$	(b) $f(x) = 1$	(c) $f(x) = 0$
d) $f(x) > 0$	(e) $f(x) \leq 0$	(f) $\lvert f(x)\rvert \leq 2$
g) $\lvert f(x)\rvert > 2$		

Lecture 3

Convergence of Sequences

3.1 Sequences

Much of our investigation of the number line and functions will be based on **sequences**, which we use to probe $\mathbb{R}$. The word "sequence" connotes a succession–that is, one object after another. Formally,

Definition 3.1. *A **sequence** of real numbers is a function from* $\mathbb{N}$ *to* $\mathbb{R}$*.*

That is, it is a list consisting of a first number, a second number, and so on. These numbers are called the *terms* of the sequence. Unless otherwise specified, a sequence is assumed to be unending–that is, there is no "last" term.[1]

Specifying a sequence

One informal way to refer to a sequence is to list its terms in order. Since there are infinitely many terms, we are actually able to list only the first few terms, leaving the rest of the sequence to the reader's imagination. This is highly unsatisfactory, as the first few terms can in principle be followed by any numbers whatsoever. For example, the sequence

$$1, 2, 3, \ldots$$

probably suggests that we are talking about the natural numbers, listed in ascending order, so the next term should be 4. However, the sequence of

[1] A list which terminates is called a **finite sequence**.

prime numbers (again listed in ascending order) also starts out with $1, 2, 3$; the next prime is 5. So we might think that the initial string

$$1, 2, 3, 5, \ldots$$

should be followed by 7. However, the sequence of *Fibonacci numbers* (where each term is the sum of the two preceding terms) follows the string $1, 2, 3, 5$, with 8, not 7. Then again, the sequence could also be a **periodic** sequence, which repeats the string $1, 2, 3, 5, 8$ over and over:

$$1, 2, 3, 5, 8, 1, 2, 3, 5, 8, 1, 2, 3, 5, 8, \ldots.$$

(And of course, in principle this string of 15 initial terms could actually be followed by anything else.)
So we need to have a better way to *specify* a sequence unambiguously: that is, we should be able to calculate or otherwise determine each term. For example, saying that we are looking at the natural numbers, or the primes, beginning with 1, in ascending order, identifies the sequence unambiguously. There are two standard ways of giving such a specification.

Closed form

The first is to say, in effect, "The k^{th} term is..." and give an explicit rule which determines each term from the number specifying its position in the sequence. This is called a **closed form** specification of the sequence. Generally, we use a subscripted notation for the terms of a sequence. The subscript is a natural number specifying the position we are referring to, called the **index** of the term. For example, we can specify the sequence of all perfect squares listed in ascending order by

$$x_k = k^2, \quad k = 1, 2, 3, \ldots.$$

An alternative shorthand for this is

$$\{k^2\}_{k=1}^{\infty}.$$

A few observations to keep in mind about this notation:

- Any letter can be used for the index, though letters from the middle of the alphabet (i, j, k, ℓ, m, n) are the conventional choice.
- The index can start at any integer, not just 1. For example, the sequence of odd natural numbers could be written as $\{2k - 1\}_{k=1}^{\infty}$, or as $\{2k + 1\}_{k=0}^{\infty}$.

- However, once the initial index is fixed, the index must proceed by increments of 1: it is *not* permissible to write the odds as $k, \quad k = 1, 3, 5, \ldots$.

Recursive definition

A second standard way to specify a sequence is to determine each term from the value of its predecessor(s) (possibly together with the index); this is called the **recursive step**. To be able to use such a definition, we also need to explicitly specify the **initial term**, or enough initial terms to allow the recursive step to be applied.
A familiar example of this is the definition of the **factorial** function $k!$, which is given by

$$\begin{aligned} 0! &= 1 \\ k! &= k \cdot (k-1)!, \quad k = 1, 2, \ldots. \end{aligned}$$

An example of a two-step recursive definition is the **Fibonacci sequence**,

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_n &= x_{n-1} + x_{n-2}, \quad n \geq 3 \end{aligned}$$

whose first few terms are

$$1, 2, 3, 5, 8, 13, 21, \ldots.$$

Series

An important class of recursively defined sequences are **series**, which are *infinite sums*. Suppose, for example, that we wish to add up the reciprocals of all the natural numbers

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Since addition is a binary operation, we can only add these up in stages: begin by adding the second reciprocal to the first

$$\frac{1}{1} + \frac{1}{2} \qquad = \frac{3}{2}$$

then add the third reciprocal to this sum

$$\frac{3}{2} + \frac{1}{3} \qquad = \frac{11}{6}$$

then in turn add the next reciprocal to this sum

$$\frac{11}{6} + \frac{1}{4} \qquad = \frac{25}{12}$$

and so on. Thus, the sequence of natural numbers

$$1, 2, 3, 4 \ldots$$

gives rise to a new sequence

$$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, ...$$

called the **partial sums** of our summation. The $\mathbf{N^{th}}$ **partial sum**

$$S_N = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$$

is more efficiently written in **summation notation**

$$S_N = \sum_{k=1}^{N} \frac{1}{k}.$$

The stepwise addition gives a recursive definition for the sequence of partial sums:

$$N = 1: \qquad S_1 = \sum_{k=1}^{1} \frac{1}{k} = \frac{1}{1} \qquad = 1$$

$$N = n+1: \quad S_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} = \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right) + \frac{1}{n+1} \quad = S_n + \frac{1}{n+1}.$$

It is the sequence of partial sums, not the original sequence of reciprocals, which determines the outcome of our calculation.

Digression:
A Caution and Clarification

It is common for beginners to confuse *sets*, *sequences*, and *series*.

- A **set** is simply a collection. It can be written as a list, but the order is irrelevant: two sets are the same if the same items are included. For example, the (finite) sets

  $$\{1,2,3\}\,,\quad \{3,2,1\}\,,\{2,1,3\}$$

  are all the same set; even if our list is repetitive, this doesn't change things: the set $\{1,2,1,3,2,1\}$ is the same as the three sets given above.

- A **sequence** (finite or infinite) is a succession of items: both order and repetition are not to be ignored. The finite **sequences**

  $$\begin{aligned} &1,2,3\\ &3,2,1\\ &2,1,3\\ &1,2,1,3,2,1 \end{aligned}$$

  are all *different*.

- A **series** is an *infinite sum*, which is a sequence of successive additions, called the **partial sums**: given an original sequence of numbers $\{a_k\}_{k=1}^{\infty}$, the sequence of partial sums is defined recursively by setting $S_1 = a_1$ and, given any particular partial sum $S_N = \sum_{k=1}^{N} a_k$, the next partial sum is

  $$S_{N+1} = \sum_{k=1}^{N+1} = S_N + a_{N+1}.$$

  In most instances, there is no closed form expression for the N^{th} partial sum.

  The standard notation for a series extends the summation notation from finite to infinite sums: given the *sequence* $\{a_k\} = a_1, a_2, a_3, \ldots$, we refer to the infinite sum of these terms as the **series**

  $$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k.$$

3.2 Mathematical Induction

A natural tool for investigating the sequence of partial sums of a series (or more generally, any recursively defined sequence) is the recursive form of argument called **mathematical induction**. The series of natural numbers furnishes a standard example of this device: the recursive definition as a series can be translated into a closed form definition of the partial sums:

Proposition 3.2. *For every* $n \in \mathbb{N}$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{3.1}$$

Proof:
(by induction:) First, Equation (**??**) holds when $n = 1$:

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2}.$$

Second, if we happen to know that Equation (**??**) holds for some particular value of $n \in \mathbb{N}$, then we can use this to determine the formula for the next term:

$$\begin{aligned}\sum_{k=1}^{n+1} k &= 1 + 2 + \cdots + n + (n+1) \\ &= \{1 + 2 + \cdots + n\} + (n+1) = \sum_{k=1}^{n} k + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

which is Equation (**??**) with n replaced by $n + 1$. □

In this argument, the first statement (that Equation (**??**) holds when $n = 1$) is called the **initial step** and the second statement (that Equation (**??**) for any particular $n \in \mathbb{N}$ implies Equation (**??**) for the next step) is called the **induction step**.

Why is this a proof of the original statement (*for all* $n \in \mathbb{N}$)? This is a "bootstrap" argument, or if you are familiar with coding, a "do" loop. For any particular value of n, say $n = k$, we automatically have a chain of implications: (the case $n = 1$) implies (the case $n = 2$) which in turn implies (the case $n = 3$), and so on, until after k iterations we arrive at: (the case $n = k - 1$) implies (the case $n = k$), as required.

3.3 Convergence

Locating Irrationals: Decimal Expansion

We saw in Theorem **??** that $\sqrt{2}$ cannot be located on the real line using the cut-and-paste strategy we used to locate rationals. How *do* we locate it?
We can use the fact that, for *positive* numbers, $x^2 < y^2$ if and only if $x < y$, to determine where $\sqrt{2}$ lies relative to any rational number. This information is recorded in the *decimal expansion* of $\sqrt{2}$.
To start, we know that $1^2 = 1 < 2 = (\sqrt{2})^2$ so $1 < \sqrt{2}$, and $(\sqrt{2})^2 = 2 < 4 = 2^2$. Thus, $\sqrt{2}$ is between 1 and 2 (and so positions $\sqrt{2}$ relative to all the *integers*). We say that the '**integer part** of $\sqrt{2}$ is $x_0 = 1$.
Having established that $\sqrt{2}$ lies between 1 and 2, we divide the interval $[1, 2]$ into 10 subintervals of equal length $\frac{1}{10}$. The left endpoints of these subintervals are $1.0, 1.1, 1.2, \ldots, 1.9$ By trial and error, we note that $1.4^2 = 1.96 < 2$ and $1.5^2 = 2.25 > 2$, so $\sqrt{2}$ lies in the subinterval with left endpoint $x_1 = 1.4$
Now we repeat this on a smaller scale: divide the interval $[1.4, 1.5]$ into ten subintervals of length $\frac{1}{10^2}$. Since $1.41^2 = 1.9881 < 2$ while $1.42^2 = 2.0164 > 2$, we know that $\sqrt{2}$ lies in the subinterval whose left endpoint is $x_2 = 1.41$. Continuing this process *ad infinitum*, we get a sequence of rational numbers x_k defined by
$x_k = \max\left\{x = \frac{p}{10^k} \,\middle|\, p \in \mathbb{N} \ \&\ x^2 < 2\right\}.$
We can mimic this process for any number $y \in \mathbb{R}$, provided we can decide where it sits relative to the fractions with denominator a power of 10. When a number is positive, this is efficiently and effectively encoded in its *decimal expansion*.[2]

[2] For a *negative* number, we use the decimal expansion of its absolute value–or what is the same thing, its negative–preceded with a minus sign.

Definition 3.3. *The* ***decimal expansion*** *of $y \in [0, \infty)$ is the sequence $\{x_n\}$ defined by*

$$x_n = \frac{p_n}{10^n} \text{ where } p_n = \max\left\{p \in \mathbb{Z} \;\middle|\; \frac{p}{10^n} \le x\right\}. \tag{3.2}$$

We refer to the n^{th} term x_n of this sequence as the ***n^{th} order decimal expansion*** *of y.*
Note that a positive rational number of the form $x = \frac{p}{10^n}$ can be written uniquely as

$$x = \frac{p}{10^n} = \lfloor x \rfloor + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \tag{3.3}$$

where

$$\lfloor x \rfloor = \max\{N \in \mathbb{Z} \mid N \le x\}$$
$$d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

$\lfloor x \rfloor$ is called the ***floor function*** *or the* ***integer part*** *of x, and the numerators d_i of the fractions in Equation (**??**) are the* ***digits*** *of the expansion; the expression in Equation (**??**) is also called the* ***decimal expansion to n digits*** *of x. A shorthand for this expression is to follow the integer part with a dot and then the n digits themselves in succession (not separated by commas)*

$$x = \lfloor x \rfloor.d_1 d_2 \ldots d_n.$$

We can use the decimal expansion of any two positive numbers to decide which is higher. Note first that since $0 = \frac{0}{10^n}$ for any $n \in \mathbb{N}$, x_n will always have the same sign as x. If two decimal expansions have the same initial string of digits up to position k and then differ in the $(k+1)^{st}$ digit, the one with the higher $(k+1)^{st}$ digit represents the larger number, which means the higher number.
Note that successive fractions with denominator 10^n are spaced $\frac{1}{10^n}$ apart, so for each $n = 0, 1, 2, \ldots$ we can say that

$$x_n \le x < x_n + \frac{1}{10^n}$$

which means in particular that the distance between x and its decimal expansion to n digits is

$$|x - x_n| < \frac{1}{10^n}.$$